# 3D Zernike Moments and Zernike Affine Invariants for 3D Image Analysis and Recognition

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**Abstract**

Guided by the results of much research work done in the past on the performance of 2D image moments and moment invariants in the presence of noise, suggesting that by using orthogonal 2D Zernike rather than regular geometrical moments one gets many advantages regarding noise effects, information suppression at low radii and redundancy, we have worked out and introduce a complete set of 3D polynomials orthonormal within the unit sphere that exhibits a "form invariance" property under 3D rotation like the 2D Zernike polynomials do in the plane. For that reason we call this set 3D Zernike polynomials. The role of the angular exponential function in the 2D Zernike polynomials set is now played by the spherical harmonics on the surface of the unit sphere. Spherical harmonics and spherical moments are introduced in a very succinct, self-contained and compact way using algebraically powerful tools like 'power substitutions' and generating functions. Unambiguous affine normalization and unique affine pose determination using 3D image moments of degree not greater than three as well as derivation of complete, uncorrelated affine invariants are naturally accomplished using the concepts we introduce in the present paper.

## 1 Introduction

Using image moments for 2D image analysis and recognition has a history of almost forty years. The major drawbacks of the ordinary or geometrical moments have been, noise sensitivity, redundancy and information suppression at low radii. To overcome these difficulties, so called Zernike moments have been developed and experimentally tested (cf. for example [1], [10], [13] and [18]) using the set of Zernike polynomials that is complete and orthonormal in the interior of the unit circle in 2D [2]. Since the experiments and the analysis have shown that 2D Zernike moments do not suffer from the abovementioned shortcomings we develop in this paper the extension of Zernike polynomials and moments to 3D in order to have also in 3D a useful all-round tool for image analysis and recognition that may be utilized for such different 3D objects like those ranging from sparse points sets to dense volumetric MRI or PET images. Some attempts of extending the theory of moment invariants to 3D have been [14] where only second degree invariants were explicitly derived and [12] where the need for invoking the representations of $SO(3)$ has been recognized but no completeness has been pursued. The need for 3D affine invariants may arise in all instances where one has to work with 3D data that have resulted from an affine reconstruction of the scene. Affine invariants will then absorb this ambiguity and can be used for recognition purposes. Further applications of 3D affine transformations are to be seen in approximately modelling non-rigid transformations, especially in conjunction with non-rigid registration algorithms. The major difficulty in developing the theory has been encountered in the treatment of the representations of the group $SO(3)$. Although there is of course a great deal of scientific literature even in conjunction with vision covering the relevant topic (cf. for example [7], [9], [11], [16] and [19]) we felt that there was still lacking some algebraically convenient tool for this subject. After some first attempt in [3] we describe in this paper the needed algebra using so called 'power substitutions' to be defined in the next section and generating functions which greatly simplifies the subject and allows the derivation of explicit algorithms given by compact formulas.

### 1.1 Power Substitutions (PS)

Given a $d$-dimensional vector $\boldsymbol{x}$ with components $x_1, x_2, \ldots, x_d$ we form all $n$-th degree monomials in the components of $\boldsymbol{x}$ and arrange them lexicographically in a new vector which we denote by $[\boldsymbol{x}]^n$ and call the $n$-th power of $\boldsymbol{x}$. It is not difficult to prove by induction that $[\boldsymbol{x}]^n$ will consist of $\binom{n+d-1}{n}$ components, i.e.

$\dim([\boldsymbol{x}]^n) = \binom{n+d-1}{n}$. Thus, any expression homogeneous in $\boldsymbol{x}$ of $n$-th degree may be written with a row vector $\boldsymbol{c}^{\boldsymbol{T}}$ of suitable dimension as $\boldsymbol{c}^{\boldsymbol{T}}[\boldsymbol{x}]^n$. Now consider some linear substitution $\boldsymbol{Ax}$ of $\boldsymbol{x}$ and let us form the $n$-th power of $\boldsymbol{Ax}$. The result of this operation will obviously have components that are homogeneous in $\boldsymbol{x}$ of $n$-th degree and hence $[\boldsymbol{Ax}]^n$ will be linear in $[\boldsymbol{x}]^n$. We denote the linear substitution lying in between with $[\boldsymbol{A}]^n$ and obtain

$$[\boldsymbol{Ax}]^n =: [\boldsymbol{A}]^n[\boldsymbol{x}]^n \ . \tag{1}$$

We call $[\boldsymbol{A}]^n$ the $n$-th power substitution ($n$-th PS) of $\boldsymbol{A}$ [15]. Note that in the relation above the matrix $\boldsymbol{A}$ may be rectangular or even a single row vector. The following properties of the $n$-th PS of matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are easily verified:

- $[\boldsymbol{AB}]^n = [\boldsymbol{A}]^n[\boldsymbol{B}]^n$
- $\left[\boldsymbol{A}^{-1}\right]^n = ([\boldsymbol{A}]^n)^{-1} =: [\boldsymbol{A}]^{-n}$ if $\boldsymbol{A}$ square regular
- $[\boldsymbol{I}]^n = \boldsymbol{I}$ where $\boldsymbol{I}$ denotes the identity matrix with appropriate dimension
- $[\lambda\boldsymbol{A}]^n = \lambda^n[\boldsymbol{A}]^n$ for some scalar $\lambda$
- $[\boldsymbol{A}]^n = [\boldsymbol{B}]^n$ iff $\boldsymbol{A} = \lambda\boldsymbol{B}$ and $\lambda^n = 1$

These relations show that $[\boldsymbol{A}]^n$ defines a representation (homomorphic map) from the algebra of matrices into itself. Further properties of the concept of power substitutions may easily be derived. We only list here some of them which are of importance in the sequel:

- $[\mathrm{diag}(\boldsymbol{a})]^n = \mathrm{diag}\left([\boldsymbol{a}]^n\right)$
- $\det\left([\boldsymbol{A}]^n\right) = (\det(\boldsymbol{A}))^{p(n,d)}$
  where $p(n,d) = \binom{n+d-1}{d}$ if $\boldsymbol{A}$ is $d \times d$ square

We note that the above defined PS concept does not handle rows and columns of $\boldsymbol{A}$ in a symmetrical manner. To find the relation between $[\boldsymbol{A}]^n$ and $\left[\boldsymbol{A}^T\right]^n$ we first determine this relationship for the case of vectors and consider to this end the $n$-th power of the inner product between $d$-dimensinal vectors $\boldsymbol{c}$ and $\boldsymbol{x}$:

$$\left(\boldsymbol{c}^T\boldsymbol{x}\right)^n = \left(\sum_{i=1}^{d} c_i x_i\right)^n = \\ = \sum_{\nu_1+\cdots+\nu_d=n} c_1^{\nu_1}\cdots c_d^{\nu_d}\frac{n!}{\nu_1!\cdots\nu_d!}x_1^{\nu_1}\cdots x_d^{\nu_d} \ .$$

If we now define a vector $\boldsymbol{p}_d^n$ containing the polynomial coefficients $\frac{n!}{\nu_1!\cdots\nu_d!}$ in lexicographic order then we may write:

$$\left(\boldsymbol{c}^T\boldsymbol{x}\right)^n = [\boldsymbol{c}]^{nT}\,\mathrm{diag}\left(\boldsymbol{p}_d^n\right)[\boldsymbol{x}]^n \ .$$

On the other hand, since $\boldsymbol{c}^T\boldsymbol{x}$ is a scalar its $n$-th power may be interpreted as the $n$-th power of a one-dimensional vector and we obtain

$$\left(\boldsymbol{c}^T\boldsymbol{x}\right)^n = \left[\boldsymbol{c}^T\boldsymbol{x}\right]^n = \left[\boldsymbol{c}^T\right]^n[\boldsymbol{x}]^n \ .$$

Comparing we arrive at

$$\left[\boldsymbol{c}^T\right]^n = [\boldsymbol{c}]^{nT}\,\mathrm{diag}(\boldsymbol{p}_d^n) \ . \tag{2}$$

Now considering the above equation for some product $\boldsymbol{Ac}$ instead of $\boldsymbol{c}$ we eventually get for a $d_1 \times d_2$ matrix $\boldsymbol{A}$ the relation:

$$\left[\boldsymbol{A}^T\right]^n = \mathrm{diag}(\boldsymbol{p}_{d_2}^n)^{-1}\,[\boldsymbol{A}]^{nT}\,\mathrm{diag}(\boldsymbol{p}_{d_1}^n) \ . \tag{3}$$

From this equation we see that if we define

$$\boldsymbol{A}^{[n]} := \sqrt{\mathrm{diag}(\boldsymbol{p}_{d_1}^n)}\,[\boldsymbol{A}]^n\,\sqrt{\mathrm{diag}(\boldsymbol{p}_{d_2}^n)^{-1}} \tag{4}$$

then we will have the symmetric result

$$\boldsymbol{A}^{T[n]} = \boldsymbol{A}^{[n]T} \ .$$

We call $\boldsymbol{A}^{[n]}$ the $n$-th symmetrized power substitution (SPS) of $\boldsymbol{A}$. Note that all properties that have been given for PS remain valid also for SPS and we additionally have:

- If $\boldsymbol{A}$ is orthogonal or unitary then so is also $\boldsymbol{A}^{[n]}$
- If $\boldsymbol{A}$ is diagonal then $\boldsymbol{A}^{[n]} = [\boldsymbol{A}]^n$

As a last remark of this section and since it is needed later we want to see what happens if we raise a vector twice in some power. In particular, we are interested in the case $\left[[\boldsymbol{\zeta}]^2\right]^l$ where $\boldsymbol{\zeta} \in \mathbb{C}^2$. Obviously, $\left[[\boldsymbol{\zeta}]^2\right]^l$ will consist of monomials in the components of $\boldsymbol{\zeta}$ of $2l$-th degree. However, since $\dim\left([\boldsymbol{\zeta}]^2\right) = 3$ we will have

$$\dim\left(\left[[\boldsymbol{\zeta}]^2\right]^l\right) = \binom{l+2}{2} > 2l+1 = \dim\left([\boldsymbol{\zeta}]^{2l}\right) \ .$$

Since $[\boldsymbol{\zeta}]^{2l}$ contains all such monomials only once there must be $\binom{l+2}{2} - (2l+1) = \binom{l}{2}$ repetitions in $\left[[\boldsymbol{\zeta}]^2\right]^l$ and we may define an $\binom{l+2}{2} \times (2l+1)$ sparse zero-one matrix $\boldsymbol{T}_l$ that contains in the $i$-th row only one nonzero entry, namely a 1 that picks from $[\boldsymbol{\zeta}]^{2l}$ the monomial that is present in the $i$-th component of $\left[[\boldsymbol{\zeta}]^2\right]^l$:

$$\left[[\boldsymbol{\zeta}]^2\right]^l =: \boldsymbol{T}_l[\boldsymbol{\zeta}]^{2l} \ . \tag{5}$$

Switching to SPS this equation reads:

$$\boldsymbol{\zeta}^{[2][l]} = \boldsymbol{V}_l\boldsymbol{\zeta}^{[2l]} \tag{6}$$

with $\boldsymbol{V}_l := \sqrt{\mathrm{diag}\left(\boldsymbol{p}_3^l\right)}\sqrt{\mathrm{diag}\left(\boldsymbol{p}_2^2\right)}^{[l]}\boldsymbol{T}_l\sqrt{\mathrm{diag}\left(\boldsymbol{p}_2^{2l}\right)}^{-1}$ again only with a single nonzero entry in each row. Since it is not difficult to infer from the *scalar* equation $\left(\boldsymbol{\zeta}^T\boldsymbol{\zeta}\right)^{[2][l]} = \left(\boldsymbol{\zeta}^T\boldsymbol{\zeta}\right)^{[2l]}$ that $\boldsymbol{V}_l^T\boldsymbol{V}_l = \boldsymbol{I}$ we also get from above

$$\boldsymbol{V}_l^T\boldsymbol{\zeta}^{[2][l]} = \boldsymbol{\zeta}^{[2l]}$$

and for any $2 \times 2$ matrix $\boldsymbol{A}$ we obtain

$$\boldsymbol{A}^{[2][l]}\boldsymbol{V}_l = \boldsymbol{V}_l\boldsymbol{A}^{[2l]} \text{ and } \boldsymbol{V}_l^T\boldsymbol{A}^{[2][l]}\boldsymbol{V}_l = \boldsymbol{A}^{[2l]} \ .$$

# 2 Moment vectors and reduction to the orthogonal case

Using the formalism introduced in the last section we can now define normalized moment vectors containing all geometrical moments of $n$-th degree (scaled appropriately as below) of any function $f(\boldsymbol{x})$ defined in the interior of the unit sphere in the following manner:

$$\boldsymbol{M}_n := \int_{|\boldsymbol{x}|\leq 1} f(\boldsymbol{x})\boldsymbol{x}^{[n]}\,d\boldsymbol{x} \Big/ \int_{|\boldsymbol{x}|\leq 1} f(\boldsymbol{x})\,d\boldsymbol{x} \ .$$

Similar expressions, as well as moment matrices

$$\boldsymbol{M}_{[p,q]} := \int_{|\boldsymbol{x}|\leq 1} f(\boldsymbol{x})\boldsymbol{x}^{[p]}\boldsymbol{x}^{[q]T}\,d\boldsymbol{x} \Big/ \int_{|\boldsymbol{x}|\leq 1} f(\boldsymbol{x})\,d\boldsymbol{x}$$

have been used in [17] for the derivation of various invariants. However, we are interested here primarily in the determination of a complete and canonical set of invariants with respect to the group of 3D affine transformations. Reduction of the affine to the orthogonal case is easily accomplished and has been repeatedly reported in the literature (cf. [6], [17], [5]). It amounts first to compute central moments and to suppose that the affinely normalized image is such that its scatter matrix $\boldsymbol{M}_{[1,1]}$ is the identity matrix, $\boldsymbol{M}_{[1,1]} = \boldsymbol{I}$. The scatter matrix $\boldsymbol{M}'_{[1,1]}$ and the moments $\boldsymbol{M}'_n$ of any image $f'(\boldsymbol{x}) = f(\boldsymbol{L}^{-1}\boldsymbol{x})$ will then read:

$$\boldsymbol{M}'_{[1,1]} = \boldsymbol{L}\boldsymbol{L}^T \text{ and} \tag{7}$$

$$\boldsymbol{M}'_n = \boldsymbol{L}^{[n]}\boldsymbol{M}_n \ . \tag{8}$$

Now, unique Cholesky decomposition of $\boldsymbol{M}'_{[1,1]}$ gives a lower triangular matrix $\boldsymbol{C}$ with $\boldsymbol{M}'_{[1,1]} = \boldsymbol{C}\boldsymbol{C}^T$ and hence we will have $\boldsymbol{C}\boldsymbol{C}^T = \boldsymbol{L}\boldsymbol{L}^T$ or equivalently $(\boldsymbol{C}^{-1}\boldsymbol{L})(\boldsymbol{C}^{-1}\boldsymbol{L})^T = \boldsymbol{I}$. That means $\boldsymbol{C}^{-1}\boldsymbol{L} = \boldsymbol{R}$ or

$$\boldsymbol{L} = \boldsymbol{C}\boldsymbol{R} \tag{9}$$

where $\boldsymbol{R}$ must be some 3D rotation/reflection, i.e. $\boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{I}$. Thus, we have already computed the linear part $\boldsymbol{L}$ of the affine transformation up to an orthogonal transformation. Equation (8) will then read

$$\boldsymbol{M}'_n = (\boldsymbol{C}\boldsymbol{R})^{[n]}\boldsymbol{M}_n \text{ or}$$

$$\boldsymbol{C}^{-[n]}\boldsymbol{M}'_n = \boldsymbol{R}^{[n]}\boldsymbol{M}_n \tag{10}$$

and we see that by multiplying each moment vector $\boldsymbol{M}'_n$ by the unique and known matrix $\boldsymbol{C}^{-[n]}$ we obtain the moment vectors of the normalized image multiplied by the SPS of some unknown orthogonal matrix $\boldsymbol{R}$. Note that $\boldsymbol{R}^{[n]}$ will then be orthogonal as well and that $\boldsymbol{C}^{-[n]}\boldsymbol{M}'_n$ will be the moment vectors of the image $f(\boldsymbol{R}^{-1}\boldsymbol{x})$ , $\boldsymbol{R} \in O(3)$. This is the essence of the reduction of the affine problem to the orthogonal case and in the next section we will have to investigate the appropriate system of basis functions in order to reach a normalization with the maximal possible economy, i.e. without losing degrees of freedom on the way of computing the invariants and exploiting as much moments of low degree as possible.

# 3 Formulation of $\boldsymbol{P} \in SO(3)$ in terms of $\boldsymbol{A} \in SU(2)$ and their irreducible representations

The study of functions or images on $\mathbb{R}^3$ under 3D rotations is greatly facilitated if we decompose the underlying function space in its irreducible (minimal) subspaces that are invariant under the action of the rotation group $SO(3)$. To derive the suitable system of basis functions we exploit the affinity of $SO(3)$ to the special unitary group $SU(2)$ i.e. the group consisting of all $2 \times 2$ complex matrices $\boldsymbol{A} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$ with $aa^* + bb^* = 1$ and hence $\boldsymbol{A}^{-1} = \boldsymbol{A}^{*T}$ , where the asterisk denotes complex conjugation. We quickly review some basic facts in elementary but explicit form.

Suppose $\boldsymbol{X}$ and $\boldsymbol{Y}$ are $2 \times 2$ complex matrices related by

$$\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{X}\boldsymbol{A}^{-1} = \boldsymbol{A}\boldsymbol{X}\boldsymbol{A}^{*T} \ . \tag{11}$$

We will then have:

- $\mathrm{tr}(\boldsymbol{X}) = \mathrm{tr}(\boldsymbol{Y})$ and $\det(\boldsymbol{X}) = \det(\boldsymbol{Y})$
- If $\boldsymbol{X}$ is hermitian then so is $\boldsymbol{Y}$: $\boldsymbol{X} = \boldsymbol{X}^{*T} \Leftrightarrow \boldsymbol{Y} = \boldsymbol{Y}^{*T}$ .

Denoting the elements of hermitian $\boldsymbol{X}$ and $\boldsymbol{Y}$ according to
$\boldsymbol{X} = \begin{pmatrix} x_3 + x_4 & x_1 + jx_2 \\ x_1 - jx_2 & -x_3 + x_4 \end{pmatrix}$ and
$\boldsymbol{Y} = \begin{pmatrix} y_3 + x_4 & y_1 + jy_2 \\ y_1 - jy_2 & -y_3 + y_4 \end{pmatrix}$ , where $j^2 = -1$
we see from $\mathrm{tr}(\boldsymbol{X}) = \mathrm{tr}(\boldsymbol{Y})$ that $y_4 = x_4$ and from $\det(\boldsymbol{X}) = \det(\boldsymbol{Y})$ that $y_4^2 - (y_1^2 + y_2^2 + y_3^2) = x_4^2 - (x_1^2 + x_2^2 + x_3^2)$ and hence also $y_1^2 + y_2^2 + y_3^2 = x_1^2 + x_2^2 + x_3^2$ . That means that (11) defines a 3D rotation $\boldsymbol{P}$ that sends the 3D real vector $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ to $\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \boldsymbol{P}\boldsymbol{x}$ and it may be verified that this concept produces all possible 3D rotations, i.e. it generates the whole $SO(3)$ group [9].[1] Now if we concatenate the rows in equation (11)

[1] In fact we get $SO(3)$ twice because $\boldsymbol{A}$ and $-\boldsymbol{A}$ yield according to (12) the same rotation.

by forming 4-vectors we obtain:

$$\mathrm{vec}(\boldsymbol{Y}) = (\boldsymbol{A} \otimes \boldsymbol{A}^*)\mathrm{vec}(\boldsymbol{X})$$

where $\otimes$ denotes the Kronecker product of matrices. With the constant matrix $\boldsymbol{T} := \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & j & 0 & 0 \\ 1 & -j & 0 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$ this equation reads

$$\boldsymbol{T}\begin{pmatrix} \boldsymbol{y} \\ y_4 \end{pmatrix} = (\boldsymbol{A} \otimes \boldsymbol{A}^*)\boldsymbol{T}\begin{pmatrix} \boldsymbol{x} \\ x_4 \end{pmatrix}$$

whence since $\boldsymbol{y} = \boldsymbol{P}\boldsymbol{x}$ and $y_4 = x_4$ we finally obtain

$$\boldsymbol{T}^{-1}(\boldsymbol{A} \otimes \boldsymbol{A}^*)\boldsymbol{T} = \begin{pmatrix} \boldsymbol{P} & \boldsymbol{0} \\ \boldsymbol{0}^T & 1 \end{pmatrix} \quad . \tag{12}$$

This equation says that the similarity transformation induced by the *constant* matrix $\boldsymbol{T}$ *simultaneously* block-diagonalizes the (reducible) representation of $SU(2)$ given by the Kronecker product $\boldsymbol{A} \otimes \boldsymbol{A}^*$ for all $\boldsymbol{A} \in SU(2)$. In the language of representation theory we say that this Kronecker product is irreducibly decomposed in a $3 \times 3$ representation given by $\boldsymbol{P}$ and in the identity representation given by 1. On the other hand it is well known that the unitary representations $\boldsymbol{A}^{[n]}$ of $SU(2)$ form a complete list of all irreducible representations of $SU(2)$ [16]. That means that $\boldsymbol{P}$ and $\boldsymbol{A}^{[2]}$ must be equivalent representations and it remains to find the similarity transformation which transforms one to the other. We obtain it by comparing (12) with the easily verified decomposition

$$\boldsymbol{Q}^{-1}(\boldsymbol{A} \otimes \boldsymbol{A}^*)\boldsymbol{Q} = \begin{pmatrix} \boldsymbol{A}^{[2]} & \boldsymbol{0} \\ \boldsymbol{0}^T & 1 \end{pmatrix} \quad , \tag{13}$$

where $\boldsymbol{Q} := \begin{pmatrix} 0 & -1/\sqrt{2} & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1/\sqrt{2} & 0 & 1 \end{pmatrix}$. The very important result reads

$$\boldsymbol{P} = \boldsymbol{U}^{-1}\boldsymbol{A}^{[2]}\boldsymbol{U} \text{ or } \boldsymbol{U}\boldsymbol{P} = \boldsymbol{A}^{[2]}\boldsymbol{U} \tag{14}$$

with $\boldsymbol{U} := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & j & 0 \\ 0 & 0 & -\sqrt{2} \\ -1 & j & 0 \end{pmatrix}$ a constant unitary $3 \times 3$ matrix.

## 3.1 Irreducible invariant subspaces and their generating functions

The subspace $L_1$ consisting of all linear functions on $\mathbb{R}^3$ is of course spanned by the components of $\boldsymbol{x}$ and a 3D rotation $\boldsymbol{P}$ is naturally acting irreducibly and invariantly on $L_1$ by $\boldsymbol{P}\boldsymbol{x}$. We consider the change of basis given by

$$\boldsymbol{e}_1(\boldsymbol{x}) = \boldsymbol{U}\boldsymbol{x} \quad .$$

Now due to (14) $\boldsymbol{P}$ is acting according to

$$\boldsymbol{e}_1(\boldsymbol{P}\boldsymbol{x}) = \boldsymbol{U}\boldsymbol{P}\boldsymbol{x} = \boldsymbol{A}^{[2]}\boldsymbol{U}\boldsymbol{x} = \boldsymbol{A}^{[2]}\boldsymbol{e}_1(\boldsymbol{x})$$

via its equivalent unitary representation $\boldsymbol{A}^{[2]}$.With $\boldsymbol{\zeta} \in \mathbb{C}^2$ we define the following generating function $e_1(\boldsymbol{x};\boldsymbol{\zeta})$ for $\boldsymbol{e}_1(\boldsymbol{x})$:

$$e_1(\boldsymbol{x};\boldsymbol{\zeta}) := \boldsymbol{\zeta}^{T[2]}\boldsymbol{e}_1(\boldsymbol{x}) \quad .$$

Note that $e_1(\boldsymbol{x};\boldsymbol{\zeta})$ is a scalar. Furthermore, it is easily seen from above that the action of a rotation $\boldsymbol{P}$ on the generating function $e_1(\boldsymbol{x};\boldsymbol{\zeta})$ is given by

$$e_1(\boldsymbol{P}\boldsymbol{x};\boldsymbol{\zeta}) = e_1(\boldsymbol{x};\boldsymbol{A}^T\boldsymbol{\zeta}) \quad . \tag{15}$$

Now we will show that we obtain irreducible invariant subspaces consisting of $l$-th degree homogeneous polynomials in $\boldsymbol{x}$ gathered together in $\boldsymbol{e}_l(\boldsymbol{x})$ simply by raising the scalar $e_1(\boldsymbol{x};\boldsymbol{\zeta})$ in the $l$-th power and by extending the definition of a generating function for $\boldsymbol{e}_1(\boldsymbol{x})$ to a generating function for $\boldsymbol{e}_l(\boldsymbol{x})$ according to

$$e_l(\boldsymbol{x};\boldsymbol{\zeta}) := \boldsymbol{\zeta}^{T[2l]}\boldsymbol{e}_l(\boldsymbol{x}) := e_1(\boldsymbol{x};\boldsymbol{\zeta})^l \quad . \tag{16}$$

Since $e_1(\boldsymbol{x};\boldsymbol{\zeta})$ is a scalar, exponentiation may also be conceived as the vector exponentiation defined in 1.1. This allows us to write:

$$\boldsymbol{\zeta}^{T[2l]}\boldsymbol{e}_l(\boldsymbol{x}) = [\boldsymbol{\zeta}^{T[2]}\boldsymbol{e}_1(\boldsymbol{x})]^{[l]} = \boldsymbol{\zeta}^{T[2][l]}\boldsymbol{e}_1(\boldsymbol{x})^{[l]}$$

and since we know from (6) that $\boldsymbol{\zeta}^{[2][l]} = \boldsymbol{V}_l\boldsymbol{\zeta}^{[2l]}$ we finally obtain from above

$$\boldsymbol{e}_l(\boldsymbol{x}) = \boldsymbol{V}_l^T\boldsymbol{e}_1(\boldsymbol{x})^{[l]} = \boldsymbol{V}_l^T\boldsymbol{U}^{[l]}\boldsymbol{x}^{[l]} \quad . \tag{17}$$

Note that the dimension of $\boldsymbol{e}_n(\boldsymbol{x})$ is $2n+1$ whereas that of $\boldsymbol{x}^{[n]}$ is $\binom{n+2}{2}$ ($\boldsymbol{V}_n$ is not square). So we don't have yet achieved a decomposition of the whole space of $n$-th degree homogeneous polynomials in $\boldsymbol{x}$. However, we will see below that this can be accomplished by augmenting the vector $\boldsymbol{e}_n(\boldsymbol{x})$ with all vectors of the form $|\boldsymbol{x}|^{2k}\boldsymbol{e}_{n-2k}(\boldsymbol{x}) = |\boldsymbol{x}|^{2k}\boldsymbol{e}_l(\boldsymbol{x})$.

The action of a rotation $\boldsymbol{P}$ on $\boldsymbol{e}_l(\boldsymbol{x})$ is best determined using generating functions. We get from (16) and (15)

$$e_l(\boldsymbol{P}\boldsymbol{x};\boldsymbol{\zeta}) = e_1(\boldsymbol{P}\boldsymbol{x};\boldsymbol{\zeta})^l = e_1(\boldsymbol{x};\boldsymbol{A}^T\boldsymbol{\zeta})^l = e_l(\boldsymbol{x};\boldsymbol{A}^T\boldsymbol{\zeta})$$

which again using the definition (16) gives

$$\boldsymbol{e}_l(\boldsymbol{P}\boldsymbol{x}) = \boldsymbol{A}^{[2l]}\boldsymbol{e}_l(\boldsymbol{x}) \quad . \tag{18}$$

The invariance of the subspace spanned by the components of $\boldsymbol{e}_l(\boldsymbol{x})$ w.r.t. 3D rotations is thus established. As we mentioned earlier the representations given by $\boldsymbol{A}^{[l]}$, $\boldsymbol{A} \in SU(2)$ are irreducible for all integer $l$. We observe that here there appear only representations with an even exponent [2l], i.e. of dimension $(2l+1)$. Now, in order to be able to give the transformation that decomposes the whole space of $n$-th degree homogeneous polynomials in its irreducible invariant subspaces we need a last definition. Since $|\boldsymbol{x}|^{2k}[\boldsymbol{x}]^l$ consists of homogeneous polynomials of $(2k+l=n)$-th degree we may define another sparse matrix $\boldsymbol{S}_{ln}$ that compiles $|\boldsymbol{x}|^{2k}[\boldsymbol{x}]^l$ from $[\boldsymbol{x}]^n$ by setting

$$|\boldsymbol{x}|^{2k}[\boldsymbol{x}]^l =: \boldsymbol{S}_{ln}[\boldsymbol{x}]^n \ .$$

Switching to SPS we obtain

$$|\boldsymbol{x}|^{2k}\boldsymbol{x}^{[l]} = \boldsymbol{S}'_{ln}\boldsymbol{x}^{[n]} \tag{19}$$

with

$$\boldsymbol{S}'_{ln} := \sqrt{\mathrm{diag}(\boldsymbol{p}_3^l)}\boldsymbol{S}_{ln}\sqrt{\mathrm{diag}(\boldsymbol{p}_3^n)^{-1}} \ .$$

We are now in a position to be able to give the square regular matrix that achieves the desired decomposition. Consider the vector

$$\boldsymbol{E}_n(\boldsymbol{x}) := \begin{bmatrix} \boldsymbol{e}_n(\boldsymbol{x}) \\ |\boldsymbol{x}|^2\boldsymbol{e}_{n-2}(\boldsymbol{x}) \\ |\boldsymbol{x}|^4\boldsymbol{e}_{n-4}(\boldsymbol{x}) \\ \vdots \\ |\boldsymbol{x}|^{n-\delta}\boldsymbol{e}_\delta(\boldsymbol{x}) \end{bmatrix}$$

where $\delta$ is 0 or 1 according to whether $n$ is even or odd respectively. We then obtain from (17) and (19)

$$\boldsymbol{E}_n(\boldsymbol{x}) = \boldsymbol{W}_n\boldsymbol{x}^{[n]} \tag{20}$$

with $\boldsymbol{W}_n = \begin{bmatrix} \boldsymbol{V}_n^T\boldsymbol{U}^{[n]} \\ \boldsymbol{V}_{n-2}^T\boldsymbol{U}^{[n-2]}\boldsymbol{S}'_{n-2,n} \\ \boldsymbol{V}_{n-4}^T\boldsymbol{U}^{[n-4]}\boldsymbol{S}'_{n-4,n} \\ \vdots \\ \boldsymbol{V}_\delta^T\boldsymbol{U}^{[\delta]}\boldsymbol{S}'_{\delta,n} \end{bmatrix}$. Now $\boldsymbol{W}_n$ is invertible and we can switch back and forth between $\boldsymbol{E}_n(\boldsymbol{x})$ and $\boldsymbol{x}^{[n]}$. The action of a rotation $\boldsymbol{P}$ is seen from (18) to be of the form

$$\boldsymbol{E}_n(\boldsymbol{P}\boldsymbol{x}) = \boldsymbol{O}_n(\boldsymbol{A})\boldsymbol{E}_n(\boldsymbol{x})$$

with $\boldsymbol{O}_n(\boldsymbol{A}) = \begin{pmatrix} \boldsymbol{A}^{[2n]} & & & \\ & \boldsymbol{A}^{[2(n-2)]} & & \\ & & \ddots & \\ & & & \boldsymbol{A}^{[2\delta]} \end{pmatrix}$ a $(2\delta+1)+(2\delta+5)+\cdots+(2n+1) = \binom{n+2}{2}$-dimensional unitary block-diagonal matrix reflecting the irreducible invariant decomposition that has been achieved. On the other hand, since we also have from (20)

$$\boldsymbol{E}_n(\boldsymbol{P}\boldsymbol{x}) = \boldsymbol{W}_n\boldsymbol{P}^{[n]}\boldsymbol{W}_n^{-1}\boldsymbol{E}_n(\boldsymbol{x})$$

we see that the similarity transformation induced by the known and *constant* matrix $\boldsymbol{W}_n$ in fact block-diagonalizes the $n$-th SPS of a rotation matrix:

$$\boldsymbol{W}_n\boldsymbol{P}^{[n]}\boldsymbol{W}_n^{-1} = \boldsymbol{O}_n(\boldsymbol{A}) \ .$$

# 4 Normalization with respect to the special orthogonal group $SO(3)$

In this section we consider the moment vectors $\boldsymbol{C}^{-[n]}\boldsymbol{M}'_n$ of an image that is offset from the normalized image $f(\boldsymbol{x})$ by some 3D rotation $\boldsymbol{P}$. Although we could carry out the normalization for general orthogonal transformations $\boldsymbol{R}$ as well (i.e. also with $\det(\boldsymbol{L}) < 0$ and therefore with $\det(\boldsymbol{R}) = -1$) again without having to use for compensation moments of higher degree than three we will confine ourselves here to $\det(\boldsymbol{L}) > 0$ and consequently $\det(\boldsymbol{R}) = 1$, i.e. in the sequel we will set $\boldsymbol{R} = \boldsymbol{P} \in SO(3)$. As we have seen in (10) the moment vectors will then relate as

$$\boldsymbol{C}^{-[n]}\boldsymbol{M}'_n = \boldsymbol{P}^{[n]}\boldsymbol{M}_n \ .$$

The transformation $\boldsymbol{W}_n$ that block-diagonalizes $\boldsymbol{P}^{[n]}$ will then decompose also the moments space in minimal subspaces that are stable under 3D rotations [5]:

$$\boldsymbol{W}_n\boldsymbol{C}^{-[n]}\boldsymbol{M}'_n = \boldsymbol{O}_n(\boldsymbol{A})\boldsymbol{W}_n\boldsymbol{M}_n \ ,$$

and if we denote the resulting subvectors of $\boldsymbol{W}_n\boldsymbol{M}_n$ and $\boldsymbol{W}_n\boldsymbol{C}^{[-n]}\boldsymbol{M}'_n$ with $\boldsymbol{m}_{nl}$ and $\boldsymbol{m}'_{nl}$ respectively, i.e.

$$\boldsymbol{W}_n\boldsymbol{M}_n = \begin{pmatrix} \boldsymbol{m}_{n,n} \\ \boldsymbol{m}_{n,n-2} \\ \vdots \\ \boldsymbol{m}_{n,\delta} \end{pmatrix}$$

and accordingly for the primed variables then we will have

$$\boldsymbol{m}'_{nl} = \boldsymbol{A}^{[2l]}\boldsymbol{m}_{nl} \ . \tag{21}$$

We call $\boldsymbol{m}_{nl}$ the spherical moments of $n$-th degree and $l$-th order. Thus, $\boldsymbol{W}_n\boldsymbol{M}_n$ contains all spherical moments of $n$-th degree with order $n$, $n-2$, $\cdots$, $\delta$.

Since $\boldsymbol{A}$ is unitary we first observe that all scalars $|\boldsymbol{m}_{nl}|$ will be invariant w.r.t. orthogonal transformations, i.e. $|\boldsymbol{m}_{nl}| = |\boldsymbol{m}'_{nl}|$. However, due to serious information loss such a system of invariants would not be complete in the sense that we could not infere from it structure information of the object. We are rather interested in an implicit normalization that is applied directly to the moments and allows the reconstruction of the image in some affinely standard position. It is expected that since the orthogonal group has 3 parameters the normalization will be achieved by fixing three degrees of freedom (d.o.f.) of the spherical moments $\boldsymbol{m}_{31}$ and $\boldsymbol{m}_{33}$. To this end we consider these moments of third degree:

$$\boldsymbol{m}'_{31} = \boldsymbol{A}^{[2]}\boldsymbol{m}_{31} \text{ and } \boldsymbol{m}'_{33} = \boldsymbol{A}^{[6]}\boldsymbol{m}_{33} \ .$$

If we denote the invariant $|\boldsymbol{m}'_{31}| = |\boldsymbol{m}_{31}|$ with $c$, $c \in \mathbb{R}^+$, and the components of $\boldsymbol{m}'_{31}/c$ with $(u, v, -u^*)$, where

$u \in \mathbb{C}$, $v \in \mathbb{R}$, $2uu^* + v^2 = 1$ and $u =: |u|e^{j\omega}$ then it is easily verified that the matrix $\boldsymbol{B}$ consisting of two unitary factors $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ of the form $\boldsymbol{B} = \boldsymbol{B}_2\boldsymbol{B}_1$ with

$$\boldsymbol{B}_1 = \begin{pmatrix} \sqrt{(1+v)/2} & -\sqrt{(1-v)/2}e^{j\omega} \\ \sqrt{(1-v)/2}e^{-j\omega} & \sqrt{(1+v)/2} \end{pmatrix}$$

and

$$\boldsymbol{B}_2 = \begin{pmatrix} e^{-j\beta} & 0 \\ 0 & e^{j\beta} \end{pmatrix}$$

is such that

$$\boldsymbol{B}^{[2]}\boldsymbol{m}'_{31} = (0, c, 0)^T =: \boldsymbol{m}_{31}$$

for any $\beta$. Thus, the factor $\boldsymbol{B}_1^{[2]}$ normalizes $\boldsymbol{m}'_{31}$ to $\boldsymbol{m}_{31} = (0, c, 0)^T$ by fixing two d.o.f. and the factor $\boldsymbol{B}_2^{[2]}$ does not affect this first partial normalization. Now we apply the transformation $\boldsymbol{B}^{[2l]}$ to all spherical moments $\boldsymbol{m}'_{nl}$ and obtain for $n = l = 3$

$$\boldsymbol{B}_2^{[6]}\boldsymbol{B}_1^{[6]}\boldsymbol{m}'_{33} =$$

$$= \text{diag}(e^{-j6\beta}, e^{-j4\beta}, e^{-j2\beta}, 1, e^{j2\beta}, e^{j4\beta}, e^{j6\beta})\boldsymbol{B}_1^{[6]}\boldsymbol{m}'_{33} \ .$$

If the third component of $\boldsymbol{B}_1^{[6]}\boldsymbol{m}'_{33}$ is denoted with $w = |w|e^{j\psi}$ then we see that we can compute $\beta$ too by demanding that the third component of $\boldsymbol{B}^{[6]}\boldsymbol{m}'_{33} = \boldsymbol{m}_{33}$ be real and positive (due to symmetry the fifth component will then be real and negative), thus fixing a third d.o.f. as expected:

$$e^{-j2\beta}e^{j\psi} = 1 \ \Rightarrow \ e^{j\beta} = \pm e^{j\psi/2} \ .$$

Note that the sign $\pm$ does not mean any ambiguity because matrices $\pm\boldsymbol{B} \in SU(2)$ translate according to (12) or (14) to the same 3D rotation. Now comparing $\boldsymbol{B}^{[2l]}\boldsymbol{m}'_{nl} = \boldsymbol{m}_{nl}$ with (21) where the computed unitary matrix $\boldsymbol{B}$ achieves the normalizations for $\boldsymbol{m}_{33}$ and $\boldsymbol{m}_{31}$ demanded above we conclude $\boldsymbol{A} = \boldsymbol{B}^{-1} = \boldsymbol{B}^{*T}$ and the linear part of the affine pose of the object $f'(\boldsymbol{x})$ with respect to the normalized object is obtained from (9) and (14):

$$\boldsymbol{L} = \boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{B}^{-[2]}\boldsymbol{U} \ .$$

The set of complete affine invariants may then be given with the aid of (8) as moments of the normalized image:

$$\boldsymbol{M}_n = \boldsymbol{L}^{-[n]}\boldsymbol{M}'_n \ . \qquad (22)$$

# 5 3D Zernike polynomials and - moments

In this section we derive a set of polynomials in the three components $x, y$ and $z$ of $\boldsymbol{x} \in \mathbb{R}^3$ which is orthonormal and complete in the unit sphere. Besides, it exhibits a certain "form invariance" with respect to 3D rotations much like the well known 2D Zernike polynomials do in the plane. The motivation is to take advantage of the many useful properties the 2D Zernike polynomials and the associated 2D Zernike moments are known to enjoy, especially when compared to the ordinary geometrical moments ([1], [10], [18]). These properties are among others: Noise insensitivity, no information suppression at low radii and no redundancy, and are naturally expected to be valid also in the 3D case. Although we will not apply these polynomials in the present paper, we would like to have them derived in order to have some standard reference in the future.

If $\boldsymbol{x} = |\boldsymbol{x}|\boldsymbol{\xi} = r\boldsymbol{\xi} = r(\sin\vartheta\cos\phi, \sin\vartheta\sin\phi, \cos\vartheta)^T$ with $|\boldsymbol{x}| = r$ and $|\boldsymbol{\xi}| = 1$ we demand for a three-fold indexed member $Z_{nl}^m(\boldsymbol{x})$ of the 3D Zernike polynomials to be of the form:

$$Z_{nl}^m(\boldsymbol{x}) = R_{nl}(r) \cdot Y_l^m(\boldsymbol{\xi}) \ ,$$

where $Y_l^m(\boldsymbol{\xi})$ are spherical harmonics of $l$-th degree orthonormal on the surface of the unit sphere with $m$ ranging from $-l$ to $l$ and $n - l$ being an even non-negative integer, $n - l =: 2k$. $R_{nl}(r)$ is the real factor depending on the radius $r$ we want to calculate so that the $Z_{nl}^m(\boldsymbol{x})$ become a set of polynomials orthonormal in the interior of the unit sphere. We collect all $2l+1$ spherical harmonics $Y_l^m(\boldsymbol{\xi})$ of $l$-th degree in a vector $\boldsymbol{Y}_l(\boldsymbol{\xi}) = (Y_l^l(\boldsymbol{\xi}), Y_l^{l-1}(\boldsymbol{\xi}), Y_l^{l-2}(\boldsymbol{\xi}), \cdots, Y_l^{-l}(\boldsymbol{\xi}))^T$ and all 3D Zernike polynomials $Z_{nl}^m(\boldsymbol{x})$ with the same indices $n$ and $l$ in a vector $\boldsymbol{Z}_{nl}(\boldsymbol{x}) = (Z_{nl}^l(\boldsymbol{x}), Z_{nl}^{l-1}(\boldsymbol{x}), Z_{nl}^{l-2}(\boldsymbol{x}), \cdots, Z_{nl}^{-l}(\boldsymbol{x}))^T$ and get

$$\boldsymbol{Z}_{nl}(\boldsymbol{x}) = R_{nl}(r)\boldsymbol{Y}_l(\boldsymbol{\xi}) \ .$$

It can be shown that $\boldsymbol{Y}_l(\boldsymbol{\xi})$ is nothing more than the vector $\boldsymbol{e}_l(\boldsymbol{\xi})$ that has been derived in section 3.1 containing $l$-th degree basis functions, appropriately scaled:

$$\boldsymbol{Y}_l(\boldsymbol{\xi}) = \sqrt{\frac{(2l+1)\binom{2l}{l}}{2^l}}\boldsymbol{e}_l(\boldsymbol{\xi}) =: k_l\boldsymbol{e}_l(\boldsymbol{\xi})$$

and the orthonormality property reads:

$$\frac{1}{4\pi}\int\limits_{|\boldsymbol{x}|=1} \boldsymbol{Y}_l(\boldsymbol{x})\boldsymbol{Y}_l(\boldsymbol{x})^{*T}\,d\boldsymbol{x} = \boldsymbol{I} \ .$$

With a 3D rotation matrix $\boldsymbol{P}$ we will then have

$$\boldsymbol{Z}_{nl}(\boldsymbol{P}\boldsymbol{x}) = R_{nl}(r)\boldsymbol{Y}_l(\boldsymbol{P}\boldsymbol{\xi}) =$$

$$= R_{nl}(r)\boldsymbol{A}^{[2l]}\boldsymbol{Y}_l(\boldsymbol{\xi}) = \boldsymbol{A}^{[2l]}\boldsymbol{Z}_{nl}(\boldsymbol{x})$$

with $\boldsymbol{A}^{[2l]}$ being the $2l$-th irreducible unitary representation of the group $SO(3)$ as we have seen in section 3.1. The relation above is the advertised "form invariance" under rotation exhibited by the polynomials $Z_{nl}^m(\boldsymbol{x})$.

In what follows we will determine the factor $R_{nl}(r)$ . Since we know that $r^lY_l^m(\boldsymbol{\xi}) = Y_l^m(\boldsymbol{x})$ is a homogeneous polynomial of $l$-th degree in the components of $\boldsymbol{x}$, it follows from $Z_{nl}^m(\boldsymbol{x}) = \frac{R_{nl}(r)}{r^l} \cdot r^l Y_l^m(\boldsymbol{\xi}) = \frac{R_{nl}(r)}{r^l} \cdot Y_l^m(\boldsymbol{x})$ that $\frac{R_{nl}(r)}{r^l}$ must be an $(n-l)/2 = k$-th degree polynomial in $r^2$ in order for $Z_{nl}^m(\boldsymbol{x})$ to be an $n$-th degree polynomial in the components of $\boldsymbol{x}$. We denote $\frac{R_{nl}(r)}{r^l}$ with $Q_{kl}(r^2)$ and obtain from the orthonormality condition of the 3D Zernike polynomials in the interior of the unit sphere

$$\frac{3}{4\pi}\int_0^1\int_0^{2\pi}\int_0^{\pi} Z_{nl}^m(\boldsymbol{x}) Z_{n'l'}^{m'}(\boldsymbol{x})^* r^2 \sin\vartheta \, d\vartheta d\phi dr = \delta_{nn'}\delta_{ll'}\delta^{mm'}$$

the condition for the polynomials $Q_{kl}$:

$$\frac{3}{2}\int_0^1 Q_{kl}(t) Q_{k'l}(t) t^{l+1/2}\, dt = \delta_{kk'} \ .$$

Thus, the $k$-th degree polynomials $Q_{kl}(t)$ may be obtained by orthogonalizing the monomials $1, t, t^2, \cdots$ with respect to the weighting factor $t^{l+1/2}$ for every $l$. Upon setting

$$Q_{kl}(t) = \sum_{\nu=0}^{k} q_{kl}^{\nu} t^{\nu}$$

we reduce the problem to the determination of the coefficients $q_{kl}^{\nu}$ for which the following equations result:

$$3\sum_{\mu=0}^{k'} q_{k'l}^{\mu} \sum_{\nu=0}^{k} \frac{q_{kl}^{\nu}}{2(\mu+\nu+l)+3} = \delta_{kk'} \ .$$

We suppose without restricting generality $k' \leq k$ and consider the equations above for all $k'$ starting from 0 to $k$. This gives the following system of equations:

$$\underbrace{\begin{pmatrix} \frac{1}{2l+3} & \frac{1}{2l+5} & \frac{1}{2l+7} & \cdots & \frac{1}{2l+2k+1} \\ \frac{1}{2l+5} & \frac{1}{2l+7} & \frac{1}{2l+9} & \cdots & \frac{1}{2l+2k+3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{2l+2k+1} & \frac{1}{2l+2k+3} & \frac{1}{2l+5} & \cdots & \frac{1}{2l+4k-1} \end{pmatrix}}_{=:\boldsymbol{A}_{kl}} \underbrace{\begin{pmatrix} q_{kl}^0 \\ q_{kl}^1 \\ \vdots \\ q_{kl}^{k-1} \end{pmatrix}}_{=:\boldsymbol{q}_{kl}}$$

$$= -q_{kl}^k \underbrace{\begin{pmatrix} \frac{1}{2l+2k+3} \\ \frac{1}{2l+2k+5} \\ \vdots \\ \frac{1}{2l+4k+1} \end{pmatrix}}_{=:\boldsymbol{a}_{kl}} \quad \text{and}$$

$$q_{kl}^k \cdot \sum_{\nu=0}^{k} \frac{q_{kl}^{\nu}}{2(l+k+\nu)+3} = \frac{1}{3}$$

which may be written compactly with the notations introduced above:

$$\boldsymbol{A}_{kl}\boldsymbol{q}_{kl} = -q_{kl}^k \cdot \boldsymbol{a}_{kl} \quad \text{and}$$

$$q_{kl}^k \cdot \left(\boldsymbol{a}_{kl}^T \boldsymbol{q}_{kl} + \frac{q_{kl}^k}{2l+4k+3}\right) = \frac{1}{3}$$

and gives the solution

$$\boldsymbol{q}_{kl} = -q_{kl}^k \boldsymbol{A}_{kl}^{-1}\boldsymbol{a}_{kl} \quad \text{with}$$

$$q_{kl}^k = \frac{1}{\sqrt{3\left[\frac{1}{2l+4k+3} - \boldsymbol{a}_{kl}^T \boldsymbol{A}_{kl}^{-1}\boldsymbol{a}_{kl}\right]}} \ .$$

Omitting the proof we also give the formula for the coefficients $q_{kl}^{\nu}$ in explicit form for all $0 \leq k$, $l$ and $0 \leq \nu \leq k$:

$$q_{kl}^{\nu} = \frac{(-1)^k}{2^{2k}}\sqrt{\frac{2l+4k+3}{3}}\binom{2k}{k}(-1)^{\nu}\frac{\binom{k}{\nu}\binom{2(k+l+\nu)+1}{2k}}{\binom{k+l+\nu}{k}} \ .$$

This result may be proved by induction and verified for any specific case testing for orthonormality with respect to the weighting function $t^{l+1/2}$. Now with known coefficients $q_{kl}^{\nu}$ the vectors of 3D Zernike polynomials are obtained in the form

$$\boldsymbol{Z}_{nl}(\boldsymbol{x}) = \sum_{\nu=0}^{k} q_{kl}^{\nu} |\boldsymbol{x}|^{2\nu} \boldsymbol{Y}_l(\boldsymbol{x}) \ .$$

We are now in a position to give explicitly the connection between the vectors of 3D Zernike moments

$$\boldsymbol{\Omega}_{nl} := \int_{|\boldsymbol{x}|\leq 1} f(\boldsymbol{x}) \boldsymbol{Z}_{nl}(\boldsymbol{x})^* \, d\boldsymbol{x} \Bigg/ \int_{|\boldsymbol{x}|\leq 1} f(\boldsymbol{x})\, d\boldsymbol{x}$$

and the scaled geometrical moments $\boldsymbol{M}_l$ of a 3D object:

$$\boldsymbol{\Omega}_{nl} = k_l \boldsymbol{V}_l^T \boldsymbol{U}^{[l]*} \sum_{\nu=0}^{k} q_{kl}^{\nu} \boldsymbol{S}'_{l,l+2\nu} \boldsymbol{M}_{l+2\nu} \ . \tag{23}$$

Finally, using in the formula above the moments of the normalized image computed in (22) we obtain the Zernike affine invariants of a 3D object in dependence of the measured moments $\boldsymbol{M}'_n$:

$$\boldsymbol{\Omega}_{nl} = k_l \boldsymbol{V}_l^T \boldsymbol{U}^{[l]*} \sum_{\nu=0}^{k} q_{kl}^{\nu} \boldsymbol{S}'_{l,l+2\nu} \boldsymbol{L}^{-[l+2\nu]} \boldsymbol{M}'_{l+2\nu} \ .$$

We know from the Weierstraß approximation theorem that any piecewise continuous function with compact region of support may be uniformly approximated by polynomials. Since we have derived a complete set of orthonormal polynomials $Z_{nl}^m(\boldsymbol{x})$ in the unit sphere and used them to build the 3D Zernike moments $\Omega_{nl}^m$, the latter are essentially generalized Fourier coefficients of $f(\boldsymbol{x})$ and if we suppose $f(\boldsymbol{x})$ piecewise continuous with region of support the interior of the unit sphere we also get a Fourier series expansion for $f(\boldsymbol{x})$:

$$f(\boldsymbol{x}) = \frac{3}{4\pi}\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/2 \rfloor}\sum_{\nu=0}^{k} q_{kl}^{\nu}|\boldsymbol{x}|^{2\nu}\sum_{m=-l}^{l}\Omega_{nl}^m Y_l^m(\boldsymbol{x})$$

with $l = n - 2k$.

This formula may be used to approximately reconstruct the shape of a general 3D object from a small number of moments of low degree simply by considering only $0 \leq n \leq N$ for some low $N$. It computes from the Zernike affine invariants the object in the standard position.

## 6 Conclusions

In this paper we have extended the theory of two-dimensional Zernike polynomials and moments to 3D in order to take advantage of the many useful properties they are known to enjoy. Complete orthonormal affine invariants that allow the reconstruction of some 3D object in an affinely standard position have resulted and we called them 3D Zernike affine invariants. We note here that if one is interested solely in 3D Euclidean invariants the presented theory is easily adapted to this case simply by leaving out the reduction to the orthogonal case part. The relevant complexity considerations can be found in [4] where we adapt a fast algorithm for computation of 2D geometrical moments [8] to the direct computation of 3D Zernike moments and invariants starting from cumulative moments of the object and without the need to go through the geometrical moments, thus avoiding the enormous dynamic range required by the latter.

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