Complete Projective Semi-Differential Invariants

Nikolaos Canterakis

Albert-Ludwigs-Universität Freiburg
Lehrstuhl für Mustererkennung und Bildverarbeitung
D-79110 Freiburg i.Br., Germany

Abstract. It is well known that given four point correspondences under perspectivity, we uniquely can determine a projectively invariant reference frame for planar scenes.
We extend this result to all semi-differential data correspondences having enough degrees of freedom to perform resection. For example, three points together with first and second curve derivatives in one of them suffice for unique frame determination. The coordinates of points in the scene with respect to this reference frame constitute a complete system of invariants. Rather than solving the Lie prolongations, we arrive at these results by exhaustively exploiting all available relationships between the given semi–differential data in two perspective views.
The generalization of the method to the 3D projective group is straightforward, thus covering the case of nonplanar scenes under stereo as well.

1 Introduction

The use of quantities that are not dependend upon the acquisition geometry (invariants) is by now well established in the area of model based vision ([3], [4]). To improve their properties in many directions the very powerful framework of semi-differential invariants has been put forward in the last few years ([8], [2]). They lie between two extremes:

(a) Invariants based on a sufficient number of known point correspondences between image and model.
(b) Differential invariants based solely on a single variable point tracing curves and on a sufficient number of derivatives at that point.

Each of these possibilities suffers from serious shortcomings: (a) entails the search for corresponding points giving rise to combinatorial explosions in search time and (b) requires the calculation of high order derivatives which is very difficult to implement, especially in the presence of noise. On the other hand, both possibilities yield complete systems of invariants, meaning that the scene is completely characterized by the invariants up to the transformation considered to be relevant from case to case (i.e. affine, projective, etc.) and against to which the invariants have been designed. Therefore, using complete systems of invariants there is no ambiguity in scene interpretation.

In contrast, semi–differential invariants need fewer points than (a) and lower order derivatives than (b), thus circumventing to a great extent the problems mentioned above. However, the systems of semi–differential invariants presented so far in the literature are not complete. This is in spite of the fact that Lie theory has been employed to underpin them theoretically [8] and that Lie methods do generally guarantee completeness of the provided solutions [6]. In fact, the differential equations emerging from applying Lie theory to the derivation of semi–differential invariants have proved too difficult to solve [8].

It is the purpose of this contribution to turn semi–differential invariants to complete systems, making out of them theoretically as well as practically, even more valuable tools for vision purposes. Rather than solving the Lie prolongations [5] we achieve this by purely algebraic methods. We will use the given semi–differential data correspondences (points and derivatives at points lying on curves) to completely determine the transformation between model and image space, assuming that the kind of this transformation is known (i.e. affine or projective). That will here be called *resection*[1]. Once this problem has been solved, every point of the model space can be put in correspondence with a point in the image space, leading to an equation which we shall call here the *transfer*[1] equation. It turns out that in all considered cases, model space data and image space data can be separated in the transfer equation in a completely symmetrical manner. This separation process leads to a vectorial equation having on both sides identical expressions, depending either on the model space or on the image space data. These expressions form for that reason invariants. Furthermore, the dependence of these vectorial invariants upon the variable point is linear in the affine case. In the projective case the homogeneous coordinates of the invariants depend linearly upon the variable point. These facts make the demonstration of completeness fairly straightforward. We will not consider in this work derivatives of the variable point. That means that semi–differential invariants of the integral type, obtained by integrating differential invariants between identifiable endpoints of a curve segment, will not appear here.

The paper is organized as follows: In Sect. 2 we fix our notation and give a formal description of the method to be applied. In Sect. 3 we consider planar scenes under affine and projective transformations. We discuss several configurations of given semi–differential data, starting from simple ones consisting only of single points and increase the order of used derivatives at identifiable curve points up to second. In Sect. 4 we consider nonplanar scenes under constrained stereo. It has been demonstrated in [1] and [7] that the formation of a so called cyclopean image vector by properly fusing the two images of a stereo setup with coplanar and aligned image planes leads either to the 3D affine or to the 3D projective group, depending on the desired degree of flexibility, regarding the parameters of the imaging system. Rather than reproducing here the geometries leading to these considerations, we go on exploring the consequences of the higher dimensionality on the invariants, proceeding much as in Sect. 3. Here, also third order derivatives will appear. Finally, Sect. 5 contains our concluding remarks.

[1] We use the terms "resection" and "transfer" in a slightly different sense than in [1] or [2] .

2 Notation and Preliminary Remarks

Fixed indexed points and the variable point of the model space will be denoted by $\boldsymbol{x}_i$ and $\boldsymbol{x}$ respectively. The corresponding points on the image space will be denoted by capital letters $\boldsymbol{X}_i$ and $\boldsymbol{X}$ respectively. In order to be able to formulate affine and projective transformations in a linear manner we introduce the augmented point vectors

$$\boldsymbol{z}_i := \begin{pmatrix} \boldsymbol{x}_i \\ 1 \end{pmatrix} , \quad \boldsymbol{z} := \begin{pmatrix} \boldsymbol{x} \\ 1 \end{pmatrix} , \quad \boldsymbol{Z}_i := \begin{pmatrix} \boldsymbol{X}_i \\ 1 \end{pmatrix} , \quad \boldsymbol{Z} := \begin{pmatrix} \boldsymbol{X} \\ 1 \end{pmatrix} \ .$$

Derivatives $\dot{\boldsymbol{x}}_i$ at curve points $\boldsymbol{x}_i$ of the model space will be computed with respect to some arbitrary parameter s which could be taken to mean arclength:

$$\dot{\boldsymbol{x}}_i := \frac{d\boldsymbol{x}_i}{ds_i} , \quad \dot{\boldsymbol{z}}_i = \begin{pmatrix} \dot{\boldsymbol{x}}_i \\ 0 \end{pmatrix} \ .$$

Since the parameterization of curves in the two spaces will in general be different, we have to introduce a new parameter for the image space for which we choose the capital letter S. Again, S could be taken to mean arclength but this is not necessary. Differentiation with respect to S will be denoted by a prime:

$$\boldsymbol{X}'_i := \frac{d\boldsymbol{X}_i}{dS_i} , \quad \boldsymbol{Z}'_i = \begin{pmatrix} \boldsymbol{X}'_i \\ 0 \end{pmatrix} \ .$$

According to this notation, differentiation of an image point $\boldsymbol{X}_i$ with respect to the *model* space parameter s_i reads as follows:

$$\dot{\boldsymbol{X}}_i = \frac{d\boldsymbol{X}_i}{ds_i} = \frac{d\boldsymbol{X}_i}{dS_i}\frac{dS_i}{ds_i} = \boldsymbol{X}'_i \cdot \dot{S}_i \qquad \text{and} \qquad \dot{\boldsymbol{Z}}_i = \boldsymbol{Z}'_i \cdot \dot{S}_i \ . \tag{1}$$

An affine transformation between the two spaces implies the relations

$$\boldsymbol{X} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{t} \qquad \text{or} \qquad \boldsymbol{Z} = \boldsymbol{B}\boldsymbol{z} \qquad \text{with} \qquad \boldsymbol{B} := \begin{pmatrix} \boldsymbol{A} & \boldsymbol{t} \\ \boldsymbol{0}^T & 1 \end{pmatrix} \ .$$

Similarly, a projective transformation reads

$$\boldsymbol{X} = \frac{\boldsymbol{A}\boldsymbol{x} + \boldsymbol{t}}{\boldsymbol{a}^T\boldsymbol{x} + 1} \quad \text{or} \quad \boldsymbol{Z} = \boldsymbol{B}\boldsymbol{z}\lambda \tag{2}$$

with $\boldsymbol{B} := \begin{pmatrix} \boldsymbol{A} & \boldsymbol{t} \\ \boldsymbol{a}^T & 1 \end{pmatrix}$ and $\lambda := \frac{1}{\boldsymbol{a}^T\boldsymbol{x}+1}$. An equivalent form of (2) will be $\boldsymbol{Z} \sim \boldsymbol{B}\boldsymbol{z}$ with $\sim$ denoting equality up to a variable scalar factor.

Our first goal will be the computation of the underlying transformation $\boldsymbol{B}$ using corresponding semi–differential data in the following form:

$$\text{(a) affine:} \qquad \boldsymbol{B} = \boldsymbol{M}^{-1}\boldsymbol{m} \tag{3}$$

$$\text{(b) projective:} \qquad \boldsymbol{B} \sim \boldsymbol{M}^{-1}\boldsymbol{m} \tag{4}$$

with $\boldsymbol{m} = f(\boldsymbol{z}_i, \dot{\boldsymbol{z}}_i, \ddot{\boldsymbol{z}}_i, \dddot{\boldsymbol{z}}_i)$ and $\boldsymbol{M} = f(\boldsymbol{Z}_i, \boldsymbol{Z}'_i, \boldsymbol{Z}''_i, \boldsymbol{Z}'''_i)$. Note that both matrices $\boldsymbol{m}$ as well as $\boldsymbol{M}$ are given in terms of exactly the same function f depending on corresponding data in exactly the same way. Of course, f will change if we move from one configuration of given data to another, or from the affine to the projective group and it is our aim to compute f for every case. For this to be feasible it is necessary that the given data configuration has at least as many degrees of freedom (d.o.f.) as the considered group. If the given configuration doesn't contain isotropies [5] algebraic manipulations will always lead, with more or less effort, to the separation (3) or (4). This will not be possible if there are isotropies that reduce the actual d.o.f. of the configuration to a smaller number than the amount of group parameters. Isotropies will manifest themselves in the form of unexpected invariants between configurations having seemingly too few d.o.f. to be able to produce any invariants at all.

Now, having decomposed $\boldsymbol{B}$ in the form (3), we get in the affine case $\boldsymbol{Z} = \boldsymbol{B}\boldsymbol{z} = \boldsymbol{M}^{-1}\boldsymbol{m}\boldsymbol{z}$ and therefore $\boldsymbol{M}\boldsymbol{Z} = \boldsymbol{m}\boldsymbol{z}$. Since both sides of this equation are identical expressions depending on corresponding data and on the variable point in the same way, it is apparent that we have constructed invariants:

$$\boldsymbol{M}\boldsymbol{Z} = \boldsymbol{m}\boldsymbol{z} =: \boldsymbol{I} \ . \tag{5}$$

Because of this relationship, we shall call $\boldsymbol{M}$ the *invariants generating matrix.* Completeness of the invariants in (5) is demonstrated by the remark that we can get back the coordinates of the variable point using the invariants and the given data in each space: $\boldsymbol{z} = \boldsymbol{m}^{-1}\boldsymbol{I}$ and $\boldsymbol{Z} = \boldsymbol{M}^{-1}\boldsymbol{I}$.

Similarly, in the projective case we get $\boldsymbol{Z} \sim \boldsymbol{B}\boldsymbol{z} \sim \boldsymbol{M}^{-1}\boldsymbol{m}\boldsymbol{z}$ and therefore

$$\boldsymbol{M}\boldsymbol{Z} \sim \boldsymbol{m}\boldsymbol{z} \sim \boldsymbol{I} \ . \tag{6}$$

This relation means that we have obtained the homogeneous coordinates of the invariants being linear in the variable point. To get true absolute invariants we now only have to divide the vector $\boldsymbol{I}$ by its last component. Again, the coordinates of the variable point in each space can be obtained using the relations $\boldsymbol{Z} \sim \boldsymbol{M}^{-1}\boldsymbol{I}$ and $\boldsymbol{z} \sim \boldsymbol{m}^{-1}\boldsymbol{I}$ and dividing the vectors on the right hand side by their last component.

Equations (5) and (6) also admit the interpretation of having used some standard reference frame. If we replace the variable point $\boldsymbol{z}$ or $\boldsymbol{Z}$ in (5) and (6) by some of the given corresponding points, we find the standard position to which this point has been mapped. In the case where the given data contain only few points and many derivatives, some of the standard positions will be occupied by points having been implicitly constructed by transformation invariant constructions contained in the product $\boldsymbol{M}\boldsymbol{Z} = \boldsymbol{m}\boldsymbol{z}$.

A word should be said about counting d.o.f. of configurations. It is clear that every given point contributes in the planar case two d.o.f. and in the nonplanar case three d.o.f. As for the derivatives, we may for simplicity assume that we are using arclength parameterization. It then follows $ds_i = |d\boldsymbol{x}_i|$ and therefore $|\dot{\boldsymbol{x}}_i| = 1$ or $(\dot{\boldsymbol{x}}_i, \dot{\boldsymbol{x}}_i) = 1$. Differentiating we obtain $(\dot{\boldsymbol{x}}_i, \ddot{\boldsymbol{x}}_i) = 0$ and $(\dot{\boldsymbol{x}}_i, \dddot{\boldsymbol{x}}_i) = -(\ddot{\boldsymbol{x}}_i, \ddot{\boldsymbol{x}}_i)$. Thus, the first derivative lying on the surface of the unit sphere contributes one d.o.f. in the planar and two d.o.f. in the nonplanar case. But so does also every other derivative since it must fulfil a scalar linear equation.

To denote specific configurations of given corresponding data we write for example ($11'1''22'3$) if we are given three points with up to second curve derivative in the first and up to first curve derivative in the second point. Point 3 may lie off the curve.

The method of arranging the given data in matrix equations which will be used throughout was inspired by [2].

3 Planar Scenes

3.1 Affine Transformations

A planar affine transformation is described by six parameters. Hence, we consider configurations of given corresponding data having six d.o.f. Every pair of corresponding points (and of course the variable points too) must fulfil the equations

$$\boldsymbol{Z} = \boldsymbol{B}\boldsymbol{z} \tag{7}$$

$$\boldsymbol{Z}'\dot{S} = \boldsymbol{B}\dot{\boldsymbol{z}} \tag{8}$$

$$\boldsymbol{Z}''\dot{S}^2 + \boldsymbol{Z}'\ddot{S} = \boldsymbol{B}\ddot{\boldsymbol{z}} \tag{9}$$

$$\boldsymbol{Z}'''\dot{S}^3 + \boldsymbol{Z}''3\dot{S}\ddot{S} + \boldsymbol{Z}'\dddot{S} = \boldsymbol{B}\dddot{\boldsymbol{z}} \tag{10}$$

which are obtained by differentiating (7) with respect to the model space parameter s and using (1). We next study some simple cases with six d.o.f. each.

Case (123). We use only (7) at every point and arrange the three obtained equations in a single matrix equation $(\boldsymbol{Z}_1\boldsymbol{Z}_2\boldsymbol{Z}_3) = \boldsymbol{B}(\boldsymbol{z}_1\boldsymbol{z}_2\boldsymbol{z}_3)$. Resection is here performed through $\boldsymbol{B} = (\boldsymbol{Z}_1\boldsymbol{Z}_2\boldsymbol{Z}_3)(\boldsymbol{z}_1\boldsymbol{z}_2\boldsymbol{z}_3)^{-1}$ and it is easy to separate model and image data in the transfer equation $\boldsymbol{Z} = (\boldsymbol{Z}_1\boldsymbol{Z}_2\boldsymbol{Z}_3)(\boldsymbol{z}_1\boldsymbol{z}_2\boldsymbol{z}_3)^{-1}\boldsymbol{z}$ obtaining $(\boldsymbol{Z}_1\boldsymbol{Z}_2\boldsymbol{Z}_3)^{-1}\boldsymbol{Z} = (\boldsymbol{z}_1\boldsymbol{z}_2\boldsymbol{z}_3)^{-1}\boldsymbol{z}$. Although this equation describes already vectorial invariants, their components are not independent since their sum must equal one. However, multiplying by the matrix $\boldsymbol{T} = \begin{pmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 1\,1\,1 \end{pmatrix}$ on the left we obtain $\underbrace{\boldsymbol{T}(\boldsymbol{Z}_1\boldsymbol{Z}_2\boldsymbol{Z}_3)^{-1}}_{\boldsymbol{M}}\boldsymbol{Z} = \underbrace{\boldsymbol{T}(\boldsymbol{z}_1\boldsymbol{z}_2\boldsymbol{z}_3)^{-1}}_{\boldsymbol{m}}\boldsymbol{z} = \boldsymbol{I}$, with $\boldsymbol{M}, \boldsymbol{m} = \begin{pmatrix} \cdot\,\cdot\,\cdot \\ \cdot\,\cdot\,\cdot \\ 0\,0\,1 \end{pmatrix}$ and $\boldsymbol{I} = (\cdot, \cdot, 1)^T$. The first two components of $\boldsymbol{I}$ form now a complete system of independent invariants. They are the components of the variable point with respect to the reference frame in which $\boldsymbol{Z}_1$ is mapped on $\boldsymbol{M}\boldsymbol{Z}_1 = (1, 0, 1)^T$, $\boldsymbol{Z}_2$ is mapped on $\boldsymbol{M}\boldsymbol{Z}_2 = (0, 1, 1)^T$ and $\boldsymbol{Z}_3$ is mapped on $\boldsymbol{M}\boldsymbol{Z}_3 = (0, 0, 1)^T$. In fact, $\boldsymbol{T}$ has been chosen so as to achieve this canonical frame.

Case $(11'22')$. This case can be reduced to the previous one by creating a third pair of corresponding points. This is here easily done by intersecting lines $(1,1')$ and $(2,2')$ in both spaces:
$z_3 := z_1 + \frac{|z_1, z_2, \dot{z}_2|}{|z_1, \dot{z}_1, \dot{z}_2|} \cdot \dot{z}_1$, $\quad Z_3 := Z_1 + \frac{|Z_1, Z_2, Z'_2|}{|Z_1, Z'_1, Z'_2|} \cdot Z'_1$. As it is well known, this is an affinely invariant construction, meaning $Z_3 = Bz_3$. The obtained invariants generating matrix reads $M = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \text{diag}\left(1, 1, \frac{|Z_1 Z'_1 Z'_2|}{|Z_1 Z_2 Z'_2|}\right) (Z_1 Z_2 Z'_1)^{-1}$.

Case $(11'1''2)$. This is the first really interesting case, in the sense that now there is no geometrically obvious affine invariant construction available for a third pair of corresponding points. However, using (7), (8) and (9) we get two equations connecting the data by arranging them in 3×3 matrices:

$$(Z_1 Z_2 Z'_1)\text{diag}(1, 1, \dot{S}_1) = B(z_1 z_2 \dot{z}_1)$$

$$(Z_1 Z'_1 Z''_1) \begin{pmatrix} 1 & & \\ & \dot{S}_1 & \ddot{S}_1 \\ & & \dot{S}_1^2 \end{pmatrix} = B(z_1 \dot{z}_1 \ddot{z}_1) \ .$$

To perform resection using the first (simpler) of the equations above we need $\dot{S}_1$. Considering only determinants and eliminating $|B|$ we can solve for $\dot{S}_1$: $\dot{S}_1 = \sqrt{\frac{|Z_1, Z_2, Z'_1|}{|Z_1, Z'_1, Z''_1|} \cdot \frac{|z_1, \dot{z}_1, \ddot{z}_1|}{|z_1, z_2, \dot{z}_1|}}$. Hence, we obtain
$B = (Z_1 Z_2 Z'_1)\text{diag}\left(1, 1, \sqrt{\left|\frac{|Z_1 Z_2 Z'_1|}{|Z_1 Z'_1 Z''_1|}\right|}\right) \text{diag}\left(1, 1, \sqrt{\left|\frac{|z_1 \dot{z}_1 \ddot{z}_1|}{|z_1 z_2 \dot{z}_1|}\right|}\right) (z_1 z_2 \dot{z}_1)^{-1}$.
Now, proceeding as in the previous cases, we get the invariants generating matrix M in the form $M = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \text{diag}\left(1, 1, \sqrt{\left|\frac{|Z_1 Z'_1 Z''_1|}{|Z_1 Z_2 Z'_1|}\right|}\right) (Z_1 Z_2 Z'_1)^{-1}$.
Comparing this result with the previous one we see that we have constructed implicitly the following third pair of points:
$z_3 = z_1 + \sqrt{\left|\frac{|z_1 z_2 \dot{z}_1|}{|z_1 \dot{z}_1 \ddot{z}_1|}\right|} \cdot \dot{z}_1$, $\quad Z_3 = Z_1 + \sqrt{\left|\frac{|Z_1 Z_2 Z'_1|}{|Z_1 Z'_1 Z''_1|}\right|} \cdot Z'_1$ and that this construction is affinely invariant, $Z_3 = Bz_3$. This is a geometrically interesting byproduct obtained purely algebraically. It involves tracing on the tangent at point 1, a line segment depending on the distance $|Z_1 Z_2 Z'_1|$ of point 2 to this tangent line and on the curvature $|Z_1 Z'_1 Z''_1|$ at point 1.

3.2 Projective Transformations

A planar projective transformation is described by eight parameters. Hence, we consider configurations of given corresponding data having eight d.o.f. Every pair of corresponding points must now obey the equations

$$Z = Bz\lambda \tag{11}$$

$$\boldsymbol{Z}'\dot{S} = \boldsymbol{B}(\boldsymbol{z}\dot{\lambda} + \dot{\boldsymbol{z}}\lambda) \tag{12}$$

$$\boldsymbol{Z}''\dot{S}^2 + \boldsymbol{Z}'\ddot{S} = \boldsymbol{B}(\boldsymbol{z}\ddot{\lambda} + 2\dot{\boldsymbol{z}}\dot{\lambda} + \ddot{\boldsymbol{z}}\lambda) \tag{13}$$

$$\boldsymbol{Z}'''\dot{S}^3 + \boldsymbol{Z}''3\dot{S}\ddot{S} + \boldsymbol{Z}'\dddot{S} = \boldsymbol{B}(\boldsymbol{z}\dddot{\lambda} + \dot{\boldsymbol{z}}3\ddot{\lambda} + \ddot{\boldsymbol{z}}3\dot{\lambda} + \dddot{\boldsymbol{z}}\lambda) \tag{14}$$

obtained by differentiating (2). The mathematics now becomes more involved because of the λ factors but the derivations follow the same lines as in the affine case. We consider some configurations having eight d.o.f. each, starting again from the simplest one.

Case (1234). We use only (11) at every given point and obtain four vector equations: $\boldsymbol{Z}_i = \boldsymbol{B}\boldsymbol{z}_i\lambda_i$, $i = 1, \cdots 4$. We can arrange them in four different matrix equations involving 3×3 matrices:

$$\begin{aligned}
(\boldsymbol{Z}_1\boldsymbol{Z}_2\boldsymbol{Z}_3) &= \boldsymbol{B}(\boldsymbol{z}_1\boldsymbol{z}_2\boldsymbol{z}_3)\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)\\
(\boldsymbol{Z}_4\boldsymbol{Z}_2\boldsymbol{Z}_3) &= \boldsymbol{B}(\boldsymbol{z}_4\boldsymbol{z}_2\boldsymbol{z}_3)\mathrm{diag}(\lambda_4, \lambda_2, \lambda_3)\\
(\boldsymbol{Z}_1\boldsymbol{Z}_4\boldsymbol{Z}_3) &= \boldsymbol{B}(\boldsymbol{z}_1\boldsymbol{z}_4\boldsymbol{z}_3)\mathrm{diag}(\lambda_1, \lambda_4, \lambda_3)\\
(\boldsymbol{Z}_1\boldsymbol{Z}_2\boldsymbol{Z}_4) &= \boldsymbol{B}(\boldsymbol{z}_1\boldsymbol{z}_2\boldsymbol{z}_4)\mathrm{diag}(\lambda_1, \lambda_2, \lambda_4) \ .
\end{aligned}$$

To perform resection using the first equation we need λ_1, λ_2 and λ_3. Considering again only determinants and eliminating $|\boldsymbol{B}|$ yields three equations for the four unknowns $\lambda_1, \lambda_2, \lambda_3$ and λ_4. However, in the projective case, $\boldsymbol{B}$ needs to be known only up to a scalar factor. That means, we may solve for λ_1, λ_2 and λ_3 and ignore some common constant factor k containing λ_4:
$\lambda_1 = \frac{|\boldsymbol{z}_4\boldsymbol{z}_2\boldsymbol{z}_3|}{|\boldsymbol{Z}_4\boldsymbol{Z}_2\boldsymbol{Z}_3|}\cdot k$, $\lambda_2 = \frac{|\boldsymbol{z}_1\boldsymbol{z}_4\boldsymbol{z}_3|}{|\boldsymbol{Z}_1\boldsymbol{Z}_4\boldsymbol{Z}_3|}\cdot k$, $\lambda_3 = \frac{|\boldsymbol{z}_1\boldsymbol{z}_2\boldsymbol{z}_4|}{|\boldsymbol{Z}_1\boldsymbol{Z}_2\boldsymbol{Z}_4|}\cdot k$ with $k = \frac{|\boldsymbol{Z}_1\boldsymbol{Z}_2\boldsymbol{Z}_3|}{|\boldsymbol{z}_1\boldsymbol{z}_2\boldsymbol{z}_3|}\cdot\lambda_4$.
Thus, we obtain the projective transformation $\boldsymbol{B}$ and the invariants generating matrix $\boldsymbol{M}$ as follows:

$$\begin{aligned}
\boldsymbol{B} \sim\ & (\boldsymbol{Z}_1\boldsymbol{Z}_2\boldsymbol{Z}_3)\mathrm{diag}(|\boldsymbol{Z}_4\boldsymbol{Z}_2\boldsymbol{Z}_3|, |\boldsymbol{Z}_1\boldsymbol{Z}_4\boldsymbol{Z}_3|, |\boldsymbol{Z}_1\boldsymbol{Z}_2\boldsymbol{Z}_4|)\cdot\\
& \cdot\mathrm{diag}(|\boldsymbol{z}_4\boldsymbol{z}_2\boldsymbol{z}_3|^{-1}, |\boldsymbol{z}_1\boldsymbol{z}_4\boldsymbol{z}_3|^{-1}, |\boldsymbol{z}_1\boldsymbol{z}_2\boldsymbol{z}_4|^{-1})(\boldsymbol{z}_1\boldsymbol{z}_2\boldsymbol{z}_3)^{-1} \ ,
\end{aligned}$$

$$\boldsymbol{M} \sim \mathrm{diag}(|\boldsymbol{Z}_4\boldsymbol{Z}_2\boldsymbol{Z}_3|^{-1}, |\boldsymbol{Z}_1\boldsymbol{Z}_4\boldsymbol{Z}_3|^{-1}, |\boldsymbol{Z}_1\boldsymbol{Z}_2\boldsymbol{Z}_4|^{-1})(\boldsymbol{Z}_1\boldsymbol{Z}_2\boldsymbol{Z}_3)^{-1} \ .$$

Now, having computed $\boldsymbol{M}$, we obtain the homogeneous coordinates of the invariant vector for any point $\boldsymbol{Z}$ by a simple multiplication:

$$\boldsymbol{I} \sim \boldsymbol{M}\boldsymbol{Z} \ . \tag{15}$$

If we elaborate this product using the specific form of $\boldsymbol{M}$, we get for the homogeneous coordinates of $\boldsymbol{I}$ ratios of signed triangle areas:

$$\boldsymbol{I} \sim \boldsymbol{M}\boldsymbol{Z} \sim \begin{pmatrix} |\boldsymbol{Z}\boldsymbol{Z}_2\boldsymbol{Z}_3|/|\boldsymbol{Z}_4\boldsymbol{Z}_2\boldsymbol{Z}_3| \\ |\boldsymbol{Z}_1\boldsymbol{Z}\boldsymbol{Z}_3|/|\boldsymbol{Z}_1\boldsymbol{Z}_4\boldsymbol{Z}_3| \\ |\boldsymbol{Z}_1\boldsymbol{Z}_2\boldsymbol{Z}|/|\boldsymbol{Z}_1\boldsymbol{Z}_2\boldsymbol{Z}_4| \end{pmatrix} \ .$$

As outlined in Sect. 2, absolute projective invariants are obtained by dividing through the last component of $\boldsymbol{I}$. Performing this operation here, we rederive nothing more than two familiar invariants, expressed in the form of cross ratios which build a complete system. However, we will prefer the more appealing form (15) exhibiting linearity in the variable point. The projective invariant frame implicitly used is easily obtained by inserting for $\boldsymbol{Z}$ in the equations above $\boldsymbol{Z}_1$, $\boldsymbol{Z}_2$, $\boldsymbol{Z}_3$ and $\boldsymbol{Z}_4$: $\boldsymbol{I}_1 \sim (100)^T, \quad \boldsymbol{I}_2 \sim (010)^T, \quad \boldsymbol{I}_3 \sim (001)^T, \quad \boldsymbol{I}_4 \sim (111)^T$. We skip the case $(11'22'3)$ since the construction $\boldsymbol{Z}_4 := (1,1') \cap (2,2')$ used in the affine case is also projectively invariant and reduces this case to the previous one.

Case (11′1″23). The equations now available are listed below:

$$(\boldsymbol{Z}_1\boldsymbol{Z}_2\boldsymbol{Z}_3) = \boldsymbol{B}(\boldsymbol{z}_1\boldsymbol{z}_2\boldsymbol{z}_3)\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3) \ ,$$

$$(\boldsymbol{Z}_1\boldsymbol{Z}_1'\boldsymbol{Z}_2)\begin{pmatrix}1 & & \\ & \dot{S}_1 & \\ & & 1\end{pmatrix} = \boldsymbol{B}(\boldsymbol{z}_1\dot{\boldsymbol{z}}_1\boldsymbol{z}_2)\begin{pmatrix}\lambda_1 & \dot{\lambda}_1 & \\ & \lambda_1 & \\ & & \lambda_2\end{pmatrix} \ ,$$

$$(\boldsymbol{Z}_1\boldsymbol{Z}_1'\boldsymbol{Z}_3)\begin{pmatrix}1 & & \\ & \dot{S}_1 & \\ & & 1\end{pmatrix} = \boldsymbol{B}(\boldsymbol{z}_1\dot{\boldsymbol{z}}_1\boldsymbol{z}_3)\begin{pmatrix}\lambda_1 & \dot{\lambda}_1 & \\ & \lambda_1 & \\ & & \lambda_3\end{pmatrix} \ ,$$

$$(\boldsymbol{Z}_1\boldsymbol{Z}_1'\boldsymbol{Z}_1'')\begin{pmatrix}1 & & \\ & \dot{S}_1 & \ddot{S}_1 \\ & & \dot{S}_1^2\end{pmatrix} = \boldsymbol{B}(\boldsymbol{z}_1\dot{\boldsymbol{z}}_1\ddot{\boldsymbol{z}}_1)\begin{pmatrix}\lambda_1 & \dot{\lambda}_1 & \ddot{\lambda}_1 \\ & \lambda_1 & 2\dot{\lambda}_1 \\ & & \lambda_1\end{pmatrix} \ .$$

To perform resection through the first equation we need λ_1, λ_2 and λ_3. The same method as before yields three equations for the four unknowns $\lambda_1, \lambda_2, \lambda_3$ and $\dot{S}_1$. Solving we obtain

$$\lambda_1 = \frac{|\boldsymbol{Z}_1\boldsymbol{Z}_1'\boldsymbol{Z}_2|^2}{|\boldsymbol{Z}_1\boldsymbol{Z}_2\boldsymbol{Z}_3|^2}\cdot\frac{|\boldsymbol{Z}_1\boldsymbol{Z}_1'\boldsymbol{Z}_3|}{|\boldsymbol{Z}_1\boldsymbol{Z}_1'\boldsymbol{Z}_1''|}\cdot\frac{|\boldsymbol{z}_1\boldsymbol{z}_2\boldsymbol{z}_3|^2}{|\boldsymbol{z}_1\dot{\boldsymbol{z}}_1\boldsymbol{z}_2|^2}\cdot\frac{|\boldsymbol{z}_1\dot{\boldsymbol{z}}_1\ddot{\boldsymbol{z}}_1|}{|\boldsymbol{z}_1\dot{\boldsymbol{z}}_1\boldsymbol{z}_3|}\cdot\lambda_3 \ ,$$

$\lambda_2 = \frac{|\boldsymbol{Z}_1\boldsymbol{Z}_1'\boldsymbol{Z}_2|}{|\boldsymbol{Z}_1\boldsymbol{Z}_1'\boldsymbol{Z}_3|}\cdot\frac{|\boldsymbol{z}_1\dot{\boldsymbol{z}}_1\boldsymbol{z}_3|}{|\boldsymbol{z}_1\dot{\boldsymbol{z}}_1\boldsymbol{z}_2|}\cdot\lambda_3$, whence the invariants generating matrix follows:

$$\boldsymbol{M} \sim \mathrm{diag}\left(\frac{|\boldsymbol{Z}_1\boldsymbol{Z}_1'\boldsymbol{Z}_2|^2}{|\boldsymbol{Z}_1\boldsymbol{Z}_2\boldsymbol{Z}_3|^2}\cdot\frac{|\boldsymbol{Z}_1\boldsymbol{Z}_1'\boldsymbol{Z}_3|}{|\boldsymbol{Z}_1\boldsymbol{Z}_1'\boldsymbol{Z}_1''|}, \frac{|\boldsymbol{Z}_1\boldsymbol{Z}_1'\boldsymbol{Z}_2|}{|\boldsymbol{Z}_1\boldsymbol{Z}_1'\boldsymbol{Z}_3|}, 1\right)(\boldsymbol{Z}_1\boldsymbol{Z}_2\boldsymbol{Z}_3)^{-1}.$$

For the projective invariant reference frame we get:
$\boldsymbol{M}\boldsymbol{Z}_1 \sim (100)^T, \quad \boldsymbol{M}\boldsymbol{Z}_2 \sim (010)^T, \quad \boldsymbol{M}\boldsymbol{Z}_3 \sim (001)^T$. As for the point mapped on $(111)^T$ we obtain the linear combination $\boldsymbol{M}^{-1}(111)^T$ of $\boldsymbol{Z}_1, \boldsymbol{Z}_2$ and $\boldsymbol{Z}_3$ with coefficients being the reciprocal values of the diagonal matrix above. Due to space limitations we must refrain from a geometrical discussion of this projectively invariant construction.

Case (11′1″22′2″). Although we count for this case eight d.o.f. too, it turns out that we cannot perform resection now. The attempt to solve this case algebraically in the same manner as all previous ones collides to a relation fulfilled by the given data only (independent of $\boldsymbol{B}$):

$$\frac{|\boldsymbol{Z}_1\boldsymbol{Z}_2\boldsymbol{Z}_2'|^3}{|\boldsymbol{Z}_1\boldsymbol{Z}_1'\boldsymbol{Z}_2|^3}\frac{|\boldsymbol{Z}_1\boldsymbol{Z}_1'\boldsymbol{Z}_1''|}{|\boldsymbol{Z}_2\boldsymbol{Z}_2'\boldsymbol{Z}_2''|} = \frac{|\boldsymbol{z}_1\boldsymbol{z}_2\dot{\boldsymbol{z}}_2|^3}{|\boldsymbol{z}_1\dot{\boldsymbol{z}}_1\boldsymbol{z}_2|^3}\frac{|\boldsymbol{z}_1\dot{\boldsymbol{z}}_1\ddot{\boldsymbol{z}}_1|}{|\boldsymbol{z}_2\dot{\boldsymbol{z}}_2\ddot{\boldsymbol{z}}_2|}.$$

This invariant which has been reported previously in the literature [2] can also be regarded as an isotropy condition, reducing the actual d.o.f. for the present case to only seven, thus making resection here impossible.

4 Non-Planar Scenes

4.1 Affine Transformations

The augmented cyclopean vectors of model and image space, $\boldsymbol{z}$ and $\boldsymbol{Z}$ respectively, are connected by the equation $\boldsymbol{Z} = \boldsymbol{B}\boldsymbol{z}$ with $\boldsymbol{B} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{t} \\ \boldsymbol{0}^T & 1 \end{pmatrix}$, where $\boldsymbol{A}$ is a 3×3 invertible matrix and $\boldsymbol{t}$ is a 3-vector. Thus, we are now faced with a group having 12 parameters and therefore we have to consider configurations with at least 12 d.o.f. The configuration (1234) with 12 d.o.f. can be treated in exactly the same way as the configuration (123) of Sect. 3, with the only difference that we have to form now 4×4 matrices. The configuration $(11'22'3)$ with 13 d.o.f. is readily reduced to (1234) by affinely invariant constructing a fourth point as the intersection of the line $(1, 1')$ and the plane $(2, 2', 3)$. Configurations $(11'1''23)$ with 13 d.o.f. and $(11'1''22')$ with 12 d.o.f. present no new difficulties.

Case $(11'1''1'''2)$. This configuration based only on two points, one of which (2) may lie off the curve, has 12 d.o.f. Using (7), (8),(9) and (10), we can arrange the given data in two 4×4 matrix equations:

$$(\boldsymbol{Z}_1\boldsymbol{Z}_2\boldsymbol{Z}_1'\boldsymbol{Z}_1'')\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \dot{S}_1 & \ddot{S}_1 \\ & & & \dot{S}_1^2 \end{pmatrix} = \boldsymbol{B}(\boldsymbol{z}_1\boldsymbol{z}_2\dot{\boldsymbol{z}}_1\ddot{\boldsymbol{z}}_1) \qquad \text{and}$$

$$(\boldsymbol{Z}_1\boldsymbol{Z}_1'\boldsymbol{Z}_1''\boldsymbol{Z}_1''')\begin{pmatrix} 1 & & & \\ & \dot{S}_1 & \ddot{S}_1 & \dddot{S}_1 \\ & & \dot{S}_1^2 & 3\dot{S}_1\ddot{S}_1 \\ & & & \dot{S}_1^3 \end{pmatrix} = \boldsymbol{B}(\boldsymbol{z}_1\dot{\boldsymbol{z}}_1\ddot{\boldsymbol{z}}_1\,\dddot{\boldsymbol{z}}_1) \ .$$

Two observations are in order: First, to perform resection using the first (simpler) of the equations above we need $\ddot{S}_1$ too. Hence, considering only determinants won't be enough. However, eliminating $\boldsymbol{B}$ yields one matrix equation the last column of which offers enough nontrivial equations to calculate $\dot{S}_1$ and $\ddot{S}_1$. Second, since the matrix containing $\dot{S}_1$ and $\ddot{S}_1$ is no more diagonal, separation is no more trivial but still feasible. Avoiding details of algebraic manipulations as well as geometrical discussions about the implicitly constructed points, we present the result in the form of the invariants generating matrix as follows:

$$\boldsymbol{M} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \operatorname{diag}(1, 1, |Q/P|^{1/3}, |Q/P|^{2/3}) \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & R/3P \\ & & & 1 \end{pmatrix} (\boldsymbol{Z}_1\boldsymbol{Z}_2\boldsymbol{Z}_1'\boldsymbol{Z}_1'')^{-1}$$

where $P := |\boldsymbol{Z}_1\boldsymbol{Z}_2\boldsymbol{Z}_1'\boldsymbol{Z}_1''|$, $Q := |\boldsymbol{Z}_1\boldsymbol{Z}_1'\boldsymbol{Z}_1''\boldsymbol{Z}_1'''|$ and $R := |\boldsymbol{Z}_1\boldsymbol{Z}_2\boldsymbol{Z}_1'\boldsymbol{Z}_1'''|$.
This result can be verified using (7), (8), (9) and (10) and checking $\boldsymbol{M}\boldsymbol{Z} = \boldsymbol{m}\boldsymbol{z}$ or through numerical simulations as we did.

4.2 Projective Transformations

The augmented cyclopean vectors of model and image space are now connected by the relation $\boldsymbol{Z} \sim \boldsymbol{B}\boldsymbol{z}$, where $\boldsymbol{B}$ is a 4×4 invertible matrix. This group has 15 parameters and we have to consider configurations with at least 15 d.o.f. Since most aspects of different cases have been already discussed at least in principle in the previous sections, we only would like to demonstrate the result of the most complicated case encountered based on two points on the curve and having 16 d.o.f.

Case (11′1″1‴22′2″). Now we can set up a system of four different matrix equations which we don't display here though. Using the abbreviations
$P := |\boldsymbol{Z}_1\boldsymbol{Z}_2\boldsymbol{Z}_1'\boldsymbol{Z}_2'|$, $Q := |\boldsymbol{Z}_1\boldsymbol{Z}_2\boldsymbol{Z}_1'\boldsymbol{Z}_1''|$, $R := |\boldsymbol{Z}_1\boldsymbol{Z}_2\boldsymbol{Z}_2'\boldsymbol{Z}_2''|$, $S := |\boldsymbol{Z}_1\boldsymbol{Z}_1'\boldsymbol{Z}_1''\boldsymbol{Z}_1'''|$,
$T := |\boldsymbol{Z}_1'\boldsymbol{Z}_2\boldsymbol{Z}_2'\boldsymbol{Z}_2''|$, $U := |\boldsymbol{Z}_1\boldsymbol{Z}_2\boldsymbol{Z}_1''\boldsymbol{Z}_2'|$, $V := |\boldsymbol{Z}_1\boldsymbol{Z}_2\boldsymbol{Z}_1'\boldsymbol{Z}_1'''|$, $W := |\boldsymbol{Z}_1\boldsymbol{Z}_1'\boldsymbol{Z}_1''\boldsymbol{Z}_2'|$
and
$C := U/P - V/(3Q), \qquad D := T/R - C, \qquad F := (QW)/(PS),$
$G := S/Q, \qquad H := (PS)/Q^2$
we first encounter the invariant $P^3S/Q^3R = p^3s/q^3r$. However, since the present configuration has 16 d.o.f. we can afford one invariant between the given corresponding data that reduces the actual d.o.f. to 15, which is what we need. The invariants generating matrix is obtained in the form

$$\boldsymbol{M} \sim \mathrm{diag}(D^{-3}, 1, D^{-2}, D^{-1}) \begin{pmatrix} 1 & C & & \\ & 1 & & F \\ & & 1 & \\ & & & 1 \end{pmatrix} \mathrm{diag}(G, 1, G, H)(\boldsymbol{Z}_1\boldsymbol{Z}_2\boldsymbol{Z}_1'\boldsymbol{Z}_2')^{-1}.$$

Again, this result can be verified using (11), (12), (13) and (14) and checking $\boldsymbol{M}\boldsymbol{Z} \sim \boldsymbol{m}\boldsymbol{z}$ or through numerical simulations as we did.

5 Conclusions

In this paper we have investigated the possibility of calculating the transformation $\boldsymbol{B}$ (affine or projective) lying between model and image space of a scene (2D or 3D), using given corresponding semi-differential data of the two spaces. It turned out that this is indeed possible if the configuration of the given data has at least as many d.o.f. as the number of parameters of the underlying group of transformations, with the proviso that there are no isotropies reducing the d.o.f. to a smaller number than the amount of parameters. In fact, isotropy prevented this process only in one case. Furthermore, we could decompose $\boldsymbol{B}$ in two matrix factors depending in a symmetrical way on the data of either space alone. Using this decomposed form of $\boldsymbol{B}$ in the transfer equation led to complete invariants depending linearly on the variable point. The invariant reference frame is determined by the invariants (standard positions) to which the given data are mapped.

We have not discussed implicitly imposed limitations on the data. But they can always be easily deduced from the algebraic expressions in which they appear. For example, in the simple case (123) of Subsection 3.1 the expression

$(\boldsymbol{z}_1\boldsymbol{z}_2\boldsymbol{z}_3)^{-1}$ is used. This matrix inversion is only possible if the vectors $\boldsymbol{z}_1$, $\boldsymbol{z}_2$ and $\boldsymbol{z}_3$ are not coplanar or, equivalently, if the points $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$ are not collinear.

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