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Algorithms for the construction of invariant features

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Abstract

The paper presents algorithms for the construction of features which are invariant with respect to a given transformation group. Two complementary approaches are considered. On the one hand we apply infinitesimal techniques (based on Lie theory) for constructing invariants for the group $GL(3, \mathbb{R})$. This has applications for 3D data sets. On the other hand we examine an alternative to the infinitesimal approach which is based on integral calculus. The basic idea is to calculate appropriate averages over the transformation group for a given function. We show how to extend these techniques to the construction of features for noncompact groups like $GL(n, \mathbb{C})$. The methods are not limited to parametric groups but can be applied to finite groups as well. However, the major advantage is that it is not necessary to solve any differential equations which is a shortcoming of many infinitesimal techniques. Explicit formulas are derived for groups which are especially important in applications. We show how to treat continuous signals in this framework. Finally we apply the presented methods to establish the existence of complete feature sets for arbitrary compact groups. In the second part we present an algorithm which eliminates successively all parameters of the full linear group $GL(3, \mathbb{R})$ acting in the space of bounded functions $\mathbb{R}^3 \to \mathbb{R}^+$ with compact support, thus yielding a complete set of affine invariant features for arbitrary compact threedimensional intensity images. The algorithm starts by computing the 3-D moments of the image up to the highest desired order and uses the so called infinitesimal method relying on the theory of Lie groups and Lie algebras. For each group parameter to be eliminated we solve a linear partial differential equation and express its solutions in terms of an integral over the image itself. By repeating this process for a special subset of group parameters in a specified order we firstly achieve a reduction of the problem to the orthogonal case. For the elimination of the remaining three parameters of the group $SO(3, \mathbb{R})$ we apply the same principle. The resulting invariants can be grouped together in subsets consisting of invariants of equal order, as the original moments do. For the computation of an invariant of n-th order we use exclusively moments up to n-th order. We obtain closed analytical, easily computable expressions for all invariants of any order. We obtain seven affine invariants of order three and $\frac{(n+1)(n+2)}{2}$ affine invariants of order n for every $n > 3$.

1 Introduction

Using features with appropriate invariance properties is an approach which has attracted considerable interest in the pattern recognition and computer vision communities during the last few years (cf. [1, 2]). A popular application of such invariant features is for recognition purposes. The seminal paper [4] demonstrated how to utilize affine invariant features of discrete point sets for model-based object recognition.

Many researchers concentrate on invariance with respect to the affine or projective group. That is natural if one is interested in general viewpoint invariance; e.g. in computer vision. However, it must be emphasized that the intended application is the decisive factor for the transformation group. For applications in quality control or visual inspection affine or projective invariance can be too general and only invariance with respect to rotations and translations may be useful. Therefore it is desirable to have algorithms to construct invariant features for arbitrary transformation groups.

We first present algorithms based on integral calculus for the construction of invariant features. The basic idea is to calculate appropriate averages over the transformation group for a given function. The corresponding mathematical techniques (the so called Haar integrals) are well established but it appears that they have not yet found the attention in the pattern recognition community they deserve. We show how to extend these techniques to the construction of features for noncompact groups like $GL(n, \mathbb{C})$.

The methods are not limited to parametric groups but can be applied to finite groups as well. However, the major advantage is that it is not necessary to solve differential equations which is a shortcoming of other techniques. Explicit formulas are derived for groups which are especially important in applications. We show how to treat continuous signals in this framework. Then we apply the presented methods to establish the existence of complete feature sets for arbitrary compact groups.

Most authors addressing the problem of constructing invariant features assume that the transformation group is a parametric group (Lie group) and apply so called infinitesimal methods from Lie group theory (cf. [14] as an excellent reference for Lie theory). That normally results in systems of differential equations which are often hard to solve. We apply these techniques for constructing invariants for the group $GL(3, \mathbb{R})$. This has applications for 3D data sets. An algorithm is presented which eliminates successively all parameters of the full linear group $GL(3, \mathbb{R})$ acting in the space of bounded functions $\mathbb{R}^3 \to \mathbb{R}^+$ with compact support, thus yielding a complete set of affine invariant features for arbitrary compact threedimensional intensity images.

We assume that the objects at hand are genuine 3D data which may have been obtained by computer tomographic reconstruction, passive 3D sensors, active range finders, stereoscopic backprojection etc. The algorithm starts by computing the 3D moments of the image up to the highest desired order and uses the infinitesimal method relying on the theory of Lie groups and Lie algebras. For each group parameter to be eliminated we solve a linear partial differential equation and express its solutions in terms of an integral over the image itself. By repeating this process for a special subset of group parameters in a specified order we firstly achieve a reduction of the problem to the orthogonal case. For the elimination of the remaining three parameters of the group $SO(3, \mathbb{R})$ we apply the same principle. However, the emerging differential equations become more and more complex. We have found that they can be simplified and solved by decomposing the function space at this stage in the smallest invariant subspaces in which $SO(3, \mathbb{R})$ is acting through its irreducible representations. This is done by using spherical harmonics.

The resulting invariants can be grouped together in subsets consisting of invariants of equal order, as the original moments do. For the computation of an invariant of n-th order we use exclusively moments up to n-th order. We obtain closed analytical, easily computable expressions for all invariants of any order. The completeness of our system of invariants becomes plausible by noting that for the elimination of each group parameter we sacrifice only one degree of freedom. We obtain seven affine invariants of order three and $\frac{(n+1)(n+2)}{2}$ affine invariants of order n for every $n > 3$.

2 Invariant Features

In this section we introduce the basic concepts and fix our notations. The signal space S is a subset of a complex vector space V. We call the elements of S patterns and denote them either by vectors $\vec{v}, \vec{w}$ or by $f(.), g(.)$ if we consider function spaces. G is a group acting via a representation on V; i.e. for every $g \in G$ exists a linear operator $\mathcal{T}(g) : V \to V$ and for these operators the following composition law is valid

$$\mathcal{T}(g_1)\mathcal{T}(g_2) = \mathcal{T}(g_1 g_2) \ \forall g_1, g_2 \in G. \tag{1}$$

Note that $g_1 g_2$ is the group product in G whereas the left-hand side of (1) denotes the product of two linear operators. The group G is called transformation group. Furthermore we assume that the signal space S is stable under the action of the group G; i.e. $\mathcal{T}(g)\vec{v} \in S \ \forall g \in G, \vec{v} \in S$.

The action of G introduces an equivalence relation $\sim$ in S. Two patterns $\vec{v}, \vec{w}$ are called equivalent, $\vec{v} \sim \vec{w}$, if a $g \in G$ exists with $\vec{v} = \mathcal{T}(g)\vec{w}$. The equivalence classes of this equivalence relation are called orbits of G in S. An orbit $\mathcal{O}(\vec{v})$ is a subset of S of the form

$$\mathcal{O}(\vec{v}) = \{\mathcal{T}(g)\vec{v} \mid g \in G\}. \tag{2}$$

Note that it is possible to generate all elements of an orbit $\mathcal{O}(\vec{v})$ by applying all group elements $g \in G$ to an arbitrary element $\vec{w} \in \mathcal{O}(\vec{v})$.

We denote by $\mathbb{C}[S]$ the set of all complex valued functions $f : S \to \mathbb{C}$ defined on the signal space S. We want to avoid a discussion of mathematical subtleties concerning the admissible class of functions. For most practical purposes it is sufficient to assume that the functions $f \in \mathbb{C}[S]$ are rational functions.

A G-invariant feature (or only feature for short) is a function $f \in \mathbb{C}[S]$ with the invariance property

$$f(\mathcal{T}(g)\vec{v}) = f(\vec{v}) \; \forall g \in G, \vec{v} \in S. \tag{3}$$

The set of all G-invariant features on the signal space S is denoted by $\mathbb{C}[S]^G$. A set of n features $f_1, f_2, \ldots, f_n$ is called complete if for $\vec{v}_1, \vec{v}_2 \in S$

$$f_i(\vec{v}_1) = f_i(\vec{v}_2) \; \forall 1 \leq i \leq n \text{ implies } \exists g \in G \text{ with } \vec{v}_1 = \mathcal{T}(g)\vec{v}_2. \tag{4}$$

The numerical values for features from a complete set coincide for two patterns if and only if these patterns are equivalent.

3 Constructing Invariant Features

3.1 Invariant Integration

A G-invariant feature is a map which has the same value for all patterns in an orbit; i.e. features are constant on orbits. Therefore they describe properties which are common to all equivalent patterns. That suggests to construct features as appropriate averages. For a given function $f \in \mathbb{C}[S]$ we try to calculate the averaged function $A[f]$ by integrating f over the orbits $\mathcal{O}(\vec{v})$:

$$A[f](\vec{v}) := \int_{\mathcal{O}(\vec{v})} f(\vec{w}) d\vec{w}. \tag{5}$$

For every $\vec{w} \in \mathcal{O}(\vec{v})$ a $g \in G$ exists such that $\vec{w} = \mathcal{T}(g)\vec{v}$. Therefore the average (5) over the orbit is equivalent to an average over the group G:

$$A[f](\vec{v}) := \int_G f(\mathcal{T}(g)\vec{v}) dg. \tag{6}$$

We call $A[f]$ the G-average (or group average) of f. We assume furthermore the following normalization condition

$$\int_G dg = 1. \tag{7}$$

Note that averaging over a group is a projection; i.e. $A(A[f]) = A[f]$. It is plain that one must impose restrictions on the group G in order to ensure the convergence of the integrals (5), (6), (7). We discuss these restrictions below and assume for the moment the existence of the integrals. Our goal is to construct G-invariant features by the integral process in (6) (i.e. $A[f] \in \mathbb{C}[S]^G$). That implies that the measure dg must obey a specific constraint. Heuristically we can derive this constraint as follows:

$$\begin{aligned} A[f](\mathcal{T}(g_1)\vec{v}) &= \int_G f(\mathcal{T}(g)\mathcal{T}(g_1)\vec{v})dg = \\ \int_G f(\mathcal{T}(gg_1)\vec{v})dg &= \int_G f(\mathcal{T}(g)\vec{v})d(gg_1^{-1}). \end{aligned}$$

Since $A[f]$ should be a feature it must possess the invariance property $A[f](\mathcal{T}(g_1)\vec{v}) = A[f](\vec{v})\ \forall g_1 \in G$. That implies for the measure dg the condition

$$dg = d(gg_1^{-1})\ \forall g_1 \in G. \tag{8}$$

Equation (8) can be interpreted intuitively as follows. The measure associates to every small region dG of the group G a 'volume element' dg. If every point in the region dG is multiplied by the group element g_1^{-1} from the right then the volume element of the 'shifted region' $dG \cdot g_1^{-1}$ is given by $d(gg_1^{-1})$. For technical reasons it is convenient to require that the measure has furthermore the invariance property $dg = d(g_1 g)\ \forall g_1 \in G$. These properties can be summarized in the statement that the volume elements must be compatible with the group structure of G.

Integrating over the transformation group as a tool for constructing invariants was invented in 1897 by the mathematician Adolf Hurwitz ([6]). The technique was generalized to the so called Haar integral (cf. [5]) and has found numerous applications in the representation theory of locally compact groups.

Note that the integral process (6) for constructing invariant features has the following advantages compared to other methods which rely on differential ('infinitesimal') techniques:

- it is not necessary to solve differential equations.
- if one is only interested in the value of $A[f]$ for a finite set of nonequivalent patterns (that is the most frequent case in model-based object recognition) it is not necessary to solve (6) analytically. Instead, one can apply routines for numerical integration.
- by appropriately partitioning the integration domain it is easy to obtain an efficient parallelization of the algorithm.

3.2 Finite and compact groups

The simplest case is that G is a finite group of order $|G|$ (order := number of elements of G). Then a group average is given by

$$A[f](\vec{v}) = \frac{1}{|G|}\sum_{g \in G} f(\mathcal{T}(g)\vec{v}). \tag{9}$$

That is applied in [7] for the extraction of translation- and rotation-invariant features from grey-scale images.

Now we assume that G is a compact topological group. Loosely speaking this means that the group elements $g \in G$ form a continuous and compact (i.e. closed and bounded) manifold in the topological sense and that the group product $g_1 \cdot g_2$ and the inversion g_1^{-1} are continuous functions of $g_1, g_2 \in G$. It is possible to prove the existence of a group average (cf. (6), (7)) for compact groups (the so called Haar integral; cf. [5]). We now derive explicit formulas for the group average for some illustrative cases.

A simple example is the planar rotation group $SO(2, \mathbb{R})$ which consists of all 2×2 matrices g of the form

$$g = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}, \ \varphi \in [0, 2\pi].$$

$SO(2, \mathbb{R})$ acts on the two-dimensional vector space $\mathbb{R}^2$ by matrix multiplication. A group average for functions $f : \mathbb{R}^2 \to \mathbb{C}$, $(v_1, v_2) \to f(v_1, v_2)$ is defined by

$$A[f](\vec{v}) = \frac{1}{2\pi} \int_{\varphi=0}^{2\pi} f\left(v_1 \cos\varphi - v_2 \sin\varphi, v_1 \sin\varphi + v_2 \cos\varphi\right) d\varphi.$$

The generalization to functions $f(\vec{v}, \vec{w}, \ldots)$ which depend upon several vectors $\vec{v}, \vec{w}, \ldots \in \mathbb{R}^2$ is straightforward.

Of special importance for the application of the invariant integral in section 3.3 is the special unitary group $SU(n, \mathbb{C})$ which is defined as the group of all $n \times n$-matrices g with complex entries and $g^t \overline{g} = I$, $\det g = 1$ (g^t is the transpose of g and $\overline{g}$ is the complex conjugate; I is the unit matrix). $SU(n, \mathbb{C})$ is a compact group ([5], p. 69). Let us derive explicitly the invariant integral for the group $SU(2, \mathbb{C})$. Every $g \in SU(2, \mathbb{C})$ may be written as (cf. [5])

$$g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \ a, b \in \mathbb{C} \text{ with } \det g = |\, a\,|^2 + |\, b\,|^2 = 1. \tag{10}$$

We write $a = x_1 + ix_2, b = x_3 + ix_4$ with real parameters $x_i, 1 \leq i \leq 4$ which gives

$$\det g = |\, a\,|^2 + |\, b\,|^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1. \tag{11}$$

Equation (11) shows that $SU(2, \mathbb{C})$ as a manifold is isomorphic to the unit sphere S_3 in $\mathbb{R}^4$. Integration over S_3 is done most effectively in four-dimensional polar coordinates:

$$\begin{aligned} x_1 &= r \sin\vartheta_1 \sin\vartheta_2 \cos\varphi \\ x_2 &= r \sin\vartheta_1 \sin\vartheta_2 \sin\varphi \end{aligned}$$

$$x_3 = r \sin\vartheta_1 \cos\vartheta_2$$
$$x_4 = r \cos\vartheta_1.$$

The equations are invertible for

$$r > 0,\ 0 < \vartheta_1, \vartheta_2 < \pi,\ 0 < \varphi < 2\pi.$$

The Jacobian $J(r, \vartheta_1, \vartheta_2, \varphi)$ for the transformation to fourdimensional polar coordinates is

$$J(r, \vartheta_1, \vartheta_2, \varphi) = r^3 \sin{\vartheta_1}^2 \sin\vartheta_2.$$

Therefore we get for the average over the group $SU(2, \mathbb{C})$ the following expressions (since we integrate over the unit sphere we can set $r = 1$):

$$\begin{aligned} dg &= \frac{1}{2\pi^2} \sin{\vartheta_1}^2 \sin\vartheta_2 d\varphi d\vartheta_1 d\vartheta_2. \\ A[f](\vec{v}) &= \int_{\varphi=0}^{2\pi} \int_{\vartheta_1=0}^{\pi} \int_{\vartheta_2=0}^{\pi} f(\mathcal{T}(g)\vec{v}) dg. \end{aligned} \tag{12}$$

3.3 Invariant Features for the general linear groups

Compact groups are the only groups of practical relevance for which the invariant integral of section 3.1 can be directly applied. However, many groups occuring in practical applications are noncompact. The most prominent examples are the groups $GL(n, \mathbb{R})$ of $n \times n$ matrices with real entries and nonvanishing determinant. They are relevant e.g. in computer vision applications for the extraction of viewpoint independent image features (cf. [2, 1]).

We will see that it is advantageous to consider the group $GL(n, \mathbb{C})$ of all $n \times n$ matrices with complex entries and nonvanishing determinant. It contains the compact group $SU(n, \mathbb{C})$ as a subgroup. For $SU(n, \mathbb{C})$ it is possible to generate features by invariant integration. We now show how to use such $SU(n, \mathbb{C})$-features for building $GL(n, \mathbb{C})$ invariants. We denote by $D(n, \mathbb{C})$ the set of all $n \times n$ diagonal matrices with complex entries and nonvanishing determinant. $D(n, \mathbb{C})$ is called dilation group. The first step is to show that invariance with respect to the groups $SU(n, \mathbb{C}), D(n, \mathbb{C})$ implies invariance with respect to the group $GL(n, \mathbb{C})$.

Lemma 1 *Let the group $G = GL(n, \mathbb{C})$ act as a transformation group on the signal space S. Let $f \in \mathbb{C}[S]$ be a function which is invariant with respect to the groups $SU(n, \mathbb{C}), D(n, \mathbb{C})$. Then f is a G-invariant feature; i.e. $f \in \mathbb{C}[S]^G$.*

That is a consequence of the polar decomposition in $G = GL(n, \mathbb{C})$ ([5], p. 28). We omit the details of the proof. Lemma 1 suggests a two step strategy for the construction of $GL(n, \mathbb{C})$-features. The first step is to calculate $SU(n, \mathbb{C})$ invariants by invariant integration. The second step is to derive from these invariants $D(n, \mathbb{C})$-features which yields by Lemma 1 automatically $GL(n, \mathbb{C})$-features. For step two no general applicable algorithm is known. However, it is often possible to get dilation invariants by forming quotients of homogeneous $SU(n, \mathbb{C})$-features (cf. below).

In order to illustrate our results we now discuss in some detail how to construct features for the group $GL(2, \mathbb{C})$. We construct features for the subgroup $SU(2, \mathbb{C})$ by using the invariant integral of equation (12). A function $f \in \mathbb{C}[S]$ is called homogeneous with degree k if

$$f(\alpha\vec{v}) = \alpha^k f(\vec{v}) \; \forall \alpha \in \mathbb{C} \setminus \{0\}, \vec{v} \in S. \tag{13}$$

Now we assume that the functions in $\mathbb{C}[S]$ are rational functions (i.e. quotients of polynomials). By integrating homogeneous polynomials $f \in \mathbb{C}[S]$ over the group $SU(2, \mathbb{C})$ we get homogeneous features for this group. The next lemma shows that the quotient of two such homogeneous polynomial $SU(2, \mathbb{C})$-features with the same degree is invariant with respect to the full group $GL(2, \mathbb{C})$.

Lemma 2 *Let $f_1, f_2 \in \mathbb{C}[S]$ be two homogeneous polynomials with the same degree which are invariant with respect to the group $SU(2, \mathbb{C})$. Then the quotient $f = \frac{f_1}{f_2}$ is a feature for the group $G = GL(2, \mathbb{C})$; $f \in \mathbb{C}[S]^G$.*

Proof: We use the parameterization (10) with the real parameters $x_i, 1 \leq i \leq 4$ for the group elements $g \in SU(2, \mathbb{C})$. Let $f_1, f_2 \in \mathbb{C}[S]$ be two homogeneous polynomials with the same degree which are invariant with respect to the group $SU(2, \mathbb{C})$. We define $f := \frac{f_1}{f_2}$. For a given $\vec{v} \in V$ we consider the following polynomial $F(\vec{x})$ (the vector $\vec{x} := (x_1, x_2, x_3, x_4)$ contains the parameters of the group elements $g \in SU(2, \mathbb{C})$):

$$F(\vec{x}) := f_1(\mathcal{T}(g)\vec{v}) \cdot f_2(\vec{v}) - f_2(\mathcal{T}(g)\vec{v}) \cdot f_1(\vec{v}).$$

Since f_1, f_2 are invariant with respect to the group $SU(2, \mathbb{C})$ we have $F(\vec{x}) = 0 \; \forall \vec{x}$ with $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$ (cf. equation 11). Due to the homogeneity and continuity of F this implies $F(\vec{x}) = 0 \; \forall \vec{x} \in \mathbb{R}^4$. Since F is a polynomial we may conclude

$$F(\vec{x}) = 0 \; \forall \vec{x} \in \mathbb{C}^4. \tag{14}$$

By an appropriate choice of $\vec{x} \in \mathbb{C}^4$ we can represent any element $g \in GL(2, \mathbb{C})$ in the form (cf. equation (10))

$$g = \begin{pmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -\bar{x_3} + i\bar{x_4} & \bar{x_1} - i\bar{x_2} \end{pmatrix}.$$

Therefore (14) implies that $f = \frac{f_1}{f_2}$ is a feature for the group $G = GL(2, \mathbb{C})$. □

Please note that it is essential here that we consider the complex groups $SU(2, \mathbb{C}), GL(2, \mathbb{C})$. The assertion of Lemma 2 is manifestly false for the real groups $SO(2, \mathbb{R}), GL(2, \mathbb{R})$.

3.4 Continuous Signals

Up to now we have assumed that the signal space is a subset of a finite dimensional vector space V. But in many applications it is more convenient to model the patterns as functions $f(\vec{x})$ depending upon the unknowns $\vec{x} = (x_1, x_2, \ldots, x_n)$. E.g. in computer vision it is a common model to assume that the measured intensity distribution is a real valued function $f(\vec{x})$ of the coordinates $\vec{x} = (x_1, x_2)$. $f(\vec{x})$ is the luminance measured at the location $\vec{x}$ in the camera plane.

In order to cope with continuous signals it is necessary to relax the assumption that V is a finite dimensional vector space. Instead, we assume that V is a Hilbert space. We denote by $\langle \ , \ \rangle$ the inner product (scalar product) in V and choose a basis $\{b_1, b_2, \ldots\} \in V$ (the basis has in general infinitely many elements). The group G acts on V by operators $\mathcal{T}(g)$:

$$\mathcal{T}(g)f = f_g, \quad f_g(\vec{x}) := f(g^{-1}\vec{x}) \ \ \forall g \in G, \ \vec{x}. \tag{15}$$

With this definition the composition law (1) holds for the operators $\mathcal{T}(g)$ (that is the reason for taking the inverse g^{-1} acting on the vector $\vec{x}$ in (15)). For $f \in V$ we denote by $m_q \in \mathbb{C}$ the inner product of f with the basis element b_q:

$$m_q := \langle f, b_q \rangle \quad \forall q = 1, 2, \ldots$$

The group action of G on V induces an action $\widetilde{\mathcal{T}}(g)$ of the group G on the coefficients m_q:

$$\widetilde{\mathcal{T}}(g) m_q = \langle \mathcal{T}(g)f, b_q \rangle = \langle f, \mathcal{T}^*(g) b_q \rangle .$$

$\mathcal{T}^*(g)$ is the adjoint operator of $\mathcal{T}(g)$. Since the $\{b_q\}$ form a basis we can write

$$\mathcal{T}^*(g) b_q = \sum_k a_{qk} b_k \ \text{ with } a_{qk} \in \mathbb{C} .$$

That yields for the group action on the coefficients m_q:

$$\widetilde{\mathcal{T}}(g) m_q = \sum_k a_{qk} \langle f, b_k \rangle = \sum_k a_{qk} m_k . \tag{16}$$

It is easy to prove by direct calculation that the composition law (1) holds for the operators $\widetilde{\mathcal{T}}(g)$.

Generally the sum on the right-hand side of (16) has infinitely many terms. That is the point to exploit the properties of the transformation group G. One tries to find a basis of V which can be split in minimal finite subsets which are stable with respect to the operators $\mathcal{T}^*(g)$:

$$\underbrace{b_1, b_2, b_3, b_4}_{\mathcal{T}^*(g)-\text{stable}}, \underbrace{b_5, b_6, b_7}_{\mathcal{T}^*(g)-\text{stable}}, \ldots$$

Stable means that the operators $\mathcal{T}^*(g)$ leave the finite dimensional subspaces spanned by the corresponding basis vectors invariant. Minimal means that these subspaces contain no further stable subsets (this property is called irreducibility). That implies e.g.

$$\mathcal{T}^*(g)b_2 = \sum_{k=1}^{4} a_{2k} b_k.$$

This induces a partition of the set of all coefficients $\{m_q\}$ which is stable with respect to the operators $\widetilde{\mathcal{T}}(g)$:

$$\underbrace{m_1, m_2, m_3, m_4}_{\widetilde{\mathcal{T}}(g)-\text{stable}}, \underbrace{m_5, m_6, m_7}_{\widetilde{\mathcal{T}}(g)-\text{stable}}, \ldots$$

E.g. for m_2:

$$\widetilde{\mathcal{T}}(g) m_2 = \langle f, \mathcal{T}^*(g) b_2 \rangle = \sum_{k=1}^{4} a_{2k} m_k;$$

i.e. the subspace spanned by m_1, m_2, m_3, m_4 is stable with respect to $\widetilde{\mathcal{T}}(g)$. Therefore the coefficients m_q are elements of finite dimensional vector spaces which are stable with respect to the transformation group G. So we have reduced the problem to the scenario of section 2 and can apply the developed methods for constructing features. Summarizing we can devise the following strategy for determining features for continuous patterns:

- fix the transformation group and the transformation law of the patterns.
- embed the patterns into a Hilbert space. That means especially to find an appropriate inner product.
- determine the induced transformation law of inner products.
- find an appropriate basis and determine the minimal finite dimensional subsets stable under the induced transformations.
- construct invariants for these finite dimensional subsets.

We mention that it is worthwile to apply this strategy in conjunction with the methods from section 3.3 to determine features for continuous signals with respect to the transformation group $GL(2, \mathbb{C})$. That results in a slight reformulation of the well known method of algebraic moments (cf. [1], pp. 375 - 397). We will discuss this issue elsewhere.

4 Complete feature sets for compact groups

In this section we establish the existence of complete feature sets (cf. equation 4) for compact groups. As shown in [3] it is sufficient for this purpose to prove that

a) the set of all G-invariant features $\mathbb{C}[S]^G$ is generated by a finite subset.

b) for any two nonequivalent patterns $\vec{v}, \vec{w} \in S$ it is possible to find a feature $f \in \mathbb{C}[S]^G$ with $f(\vec{v}) \neq f(\vec{w})$. In this case S is called separable.

$\mathbb{C}[S]^G$ is finitely generated since representations of compact groups are completely reducible (cf. [5, 3]). In the next Lemma we prove that the pattern space is separable for any compact group. Furthermore we will see that it is possible to find complete sets of polynomial features.

Lemma 3 *Let G be a compact group acting on V as a transformation group. Then the pattern space S is separable and it is possible to construct a complete set of polynomial features.*

Proof: Since $\mathbb{C}[S]^G$ is finitely generated it is sufficient to prove that for any two nonequivalent patterns $\vec{v}, \vec{w} \in S$ a polynomial feature $f \in \mathbb{C}[S]^G$ exists with $f(\vec{v}) \neq f(\vec{w})$. We sketch the main argument from ([5], p.281). Denote by $\| \, . \, \|$ the norm derived from the inner product in V. Define a map $d_{\vec{v},\vec{w}} : S \rightarrow \mathbb{R}$ by

$$d_{\vec{v},\vec{w}}(\vec{k}) := \min_{g \in G} \left(\| \vec{k} - \mathcal{T}(g)\vec{v} \| \right) - \min_{g \in G} \left(\| \vec{k} - \mathcal{T}(g)\vec{w} \| \right).$$

Since G is compact and every continuous function assumes its extrema on compact sets an $\varepsilon \in \mathbb{R}, \varepsilon > 0$ exists so that the function $d_{\vec{v},\vec{w}}(\vec{k})$ has the properties:

$$\begin{aligned} d_{\vec{v},\vec{w}}(\vec{k}) &> 0 \ \forall \vec{k} \in \mathcal{O}(\vec{w}). \\ d_{\vec{v},\vec{w}}(\vec{k}) &< 0 \ \forall \vec{k} \in \mathcal{O}(\vec{v}). \\ | \, d_{\vec{v},\vec{w}}(\vec{k}) \, | &\geq \varepsilon \ \forall \vec{k} \in \mathcal{O}(\vec{v}), \mathcal{O}(\vec{w}). \end{aligned}$$

Note that $d_{\vec{v},\vec{w}}(\vec{k})$ is generally not a polynomial. But all involved sets are compact. Due to the Weierstraß approximation theorem it is therefore possible to find a polynomial $p \in \mathbb{C}[S]$ with $| d_{\vec{v},\vec{w}}(\vec{k}) - p(\vec{k}) | < \varepsilon \ \forall \vec{k} \in \mathcal{O}(\vec{v}), \mathcal{O}(\vec{w})$. The polynomial p has the properties:

$$\begin{aligned} p(\vec{k}) &> 0 \ \forall \vec{k} \in \mathcal{O}(\vec{w}). \\ p(\vec{k}) &< 0 \ \forall \vec{k} \in \mathcal{O}(\vec{v}). \end{aligned}$$

By integrating the polynomial p over the group G we get a feature $f = A[p] \in \mathbb{C}[S]^G$ with $f(\vec{v}) < 0$ and $f(\vec{w}) > 0$. □

5 A complete set of affine invariant features for 3D intensity images

5.1 Introduction

In this part of the paper we consider the concrete pattern recognition problem obtained by setting

$$\begin{aligned} S &= \{f(\boldsymbol{x}) : I\!R^3 \to I\!R^+ \mid f(\boldsymbol{x}) < \infty,\ f(\boldsymbol{x}) = 0\ \forall\ |\boldsymbol{x}| \geq \lambda > 0,\ \ \lambda < \infty\} \\ G &= GA(3, I\!R). \end{aligned}$$

We denote by $GA(3, I\!R)$ the group of affine transformations on a threedimensional real Euclidean space. A group element $g \in G$ is characterized by an arbitrary real 3×3 matrix $\boldsymbol{A}$ with nonvanishing determinant and a real threedimensional vector $\boldsymbol{b}$. The group action of G in S is given by

$$\mathcal{T}(g)f(\boldsymbol{x}) = f(\boldsymbol{A}^{-1}\boldsymbol{x} - \boldsymbol{b}).$$

For the solution of this problem (albeit in an n-dim. space) it is first shown in [9] how to accomplish a reduction of the action of the affine group to that of the orthogonal group. To this end moments up to second order are computed. They are used to define a transformation which is then applied to the actual pattern. The result is a pattern which differs from a normalized reference pattern by an unknown orthogonal transformation. Invariants are then derived by contracting indices of the new moment tensor. Thus, it is necessary to compute image moments twice plus an additional affine transformation to the full image. The resulting invariants are by no means complete and may even be not independent. Similar remarks apply also for tensor and matrix techniques used in [10] and [11] respectively.

For the computation of invariants of threedimensional images with respect to the orthogonal group $SO(3)$ the Clebsch-Gordan coefficients of tensor product representations are used in [12]. Various invariants of low order (≤ 3) are derived without pursuing completeness.

Instead, we compute the image moments up to a highest desired order once. This can most effectively be done by using a straightforward 3D generalization of a fast 2D moment generating algorithm for grey scale images [13]. Starting with the moments we define in several steps intermediate variables and achieve also a reduction of the problem to the orthogonal group. The algorithm used in each step is the result of solving a linear partial differential equation obtained by applying the theory of Lie groups and Lie algebras. The same procedure is then put forward in the irreducible invariant subspaces of the group $SO(3)$.

5.2 Preliminary Remarks

It is well known that for functions $f(\boldsymbol{x}) \in S$ as above all moments

$$M^{pqr} := \int_{\mathbb{R}^3} f(\boldsymbol{x}) x^p y^q z^r \, d\boldsymbol{x} \quad ; \quad \boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

exist and that due to the Weierstraß approximation theorem they uniquely define a piecewise continuous function $f \in S$. We can therefore use moments as an adequate description of images. Since we can at once derive central moments as well as divide all moments by the zeroth moment we may consider only images for which

$$\begin{aligned} M^{000} &= \int_{\mathbb{R}^3} f(\boldsymbol{x}) \, d\boldsymbol{x} = 1 \qquad \text{and} \\ \boldsymbol{M}_1 &:= \begin{pmatrix} M^{100} \\ M^{010} \\ M^{001} \end{pmatrix} = \int_{\mathbb{R}^3} f(\boldsymbol{x}) \boldsymbol{x} \, d\boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

is valid. This amounts to a reduction of the affine group to the general linear group $GL(3, \mathbb{R})$ with the group action

$$\mathcal{T}(g) f(\boldsymbol{x}) = \frac{1}{|\boldsymbol{A}|} f(\boldsymbol{A}^{-1}\boldsymbol{x}) =: f_A(\boldsymbol{x})$$

where

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} =: \begin{pmatrix} \boldsymbol{a}_1^{\mathrm{T}} \\ \boldsymbol{a}_2^{\mathrm{T}} \\ \boldsymbol{a}_3^{\mathrm{T}} \end{pmatrix}$$

$$\det(\boldsymbol{A}) = \boldsymbol{a}_1^{\mathrm{T}}(\boldsymbol{a}_2 \times \boldsymbol{a}_3) \neq 0$$

and

$$|\boldsymbol{A}| := \mathrm{abs}(\det(\boldsymbol{A})).$$

We can in principle represent an orbit

$$\mathcal{O}(f) = \{ f_A(\boldsymbol{x}) \in S \quad | \quad \boldsymbol{A} \in GL(3, \mathbb{R}) \}$$

by any of its elements. However, we choose as representative reference pattern the one meeting certain conditions. Six conditions concern moments of second order and are formulated with the aid of the covariance matrix:

$$\boldsymbol{\Sigma}_M := \int_{\mathbb{R}^3} f(\boldsymbol{x}) \boldsymbol{x} \boldsymbol{x}^{\mathrm{T}} \, d\boldsymbol{x} = \begin{pmatrix} M^{200} & M^{110} & M^{101} \\ M^{110} & M^{020} & M^{011} \\ M^{101} & M^{011} & M^{002} \end{pmatrix} = \boldsymbol{I} \tag{17}$$

where $\boldsymbol{I}$ denotes the 3×3 unit matrix. Further conditions concerning moments of third order can be derived from the elimination algorithm. However, we don't need them here. We will denote such normalized reference patterns without subscript, e.g. $f(\boldsymbol{x})$.

Our aim shall be the derivation of a coordinate system for the function space S in which the coordinates of $f_A(\boldsymbol{x})$ split in two parts:
a) between - orbits coordinates
b) within - orbit coordinates.
The between - orbits coordinates $J_i: \ S \rightarrow \mathbb{C}, i \in \mathbb{N}$ should be invariant within every specific orbit as well as uniquely characterize it:

$$J_i(f_A(\boldsymbol{x})) = J_i(f(\boldsymbol{x})) \quad \forall \quad \boldsymbol{A}, i$$

$$J_i(h(\boldsymbol{x})) = J_i(f(\boldsymbol{x})) \quad \forall \quad i \quad \Leftrightarrow \quad h \sim f.$$

We readily recognize the set $\mathbb{I} := \{J_i \mid i \in \mathbb{N}\}$ as a complete set of invariant features for the action of $GL(3, \mathbb{R})$ in S. It is possible through straightforward manipulations to rebuild the set $\mathbb{I}$ in such a way as to obtain the coordinates of the reference pattern with respect to an orthonormal system of polynomial basis functions. This orthogonalization process will lead to a canonical set of invariant features. As within-orbit coordinates of $f_A(\boldsymbol{x})$ we can define the parameters of the matrix $\boldsymbol{A}$ which transforms the reference pattern $f(\boldsymbol{x})$ to $f_A(\boldsymbol{x})$. They can be uniquely derived using moments of $f_A(\boldsymbol{x})$ up to third order. However, we discuss here only a) without going into the process of orthogonalizing the invariants.

5.3 Reduction to the orthogonal group

As already mentioned we start with the moments of an actual image $f_A(\boldsymbol{x})$. Neither $f(\boldsymbol{x})$ nor $\boldsymbol{A}$ can be assumed to be known:

$$M_A^{pqr} = \int_{\mathbb{R}^3} f_A(\boldsymbol{x}) x^p y^q z^r \, d\boldsymbol{x} = \tag{18}$$

$$= \int_{\mathbb{R}^3} f(\boldsymbol{x}) (\boldsymbol{a}_1^{\mathrm{T}} \boldsymbol{x})^p (\boldsymbol{a}_2^{\mathrm{T}} \boldsymbol{x})^q (\boldsymbol{a}_3^{\mathrm{T}} \boldsymbol{x})^r \, d\boldsymbol{x}. \tag{19}$$

From the equation above we immediatelly recognize the subset of all moments of equal order as being stable under $GL(3, \mathbb{R})$. For the covariance matrix consisting of moments of second order we get:

$$\boldsymbol{\Sigma}_{M_A} = \int_{\mathbb{R}^3} f_A(\boldsymbol{x}) \boldsymbol{x} \boldsymbol{x}^{\mathrm{T}} \, d\boldsymbol{x} =$$

$$= \int_{\mathbb{R}^3} f(\boldsymbol{x}) \boldsymbol{A} \boldsymbol{x} \boldsymbol{x}^{\mathrm{T}} \boldsymbol{A}^{\mathrm{T}} \, d\boldsymbol{x} = \boldsymbol{A} \int_{\mathbb{R}^3} f(\boldsymbol{x}) \boldsymbol{x} \boldsymbol{x}^{\mathrm{T}} \, d\boldsymbol{x} \ \boldsymbol{A}^{\mathrm{T}}$$

and due to (17)

$$\boldsymbol{\Sigma}_{M_A} = \boldsymbol{A} \boldsymbol{A}^{\mathrm{T}}.$$

We seek functions J depending on moments M_A^{pqr} for which the invariance property

$$J(\cdots M_A^{pqr}\cdots) \equiv J(\cdots M_I^{pqr}\cdots) \tag{20}$$

is valid. In the sequel we will drop the subscript I: $M^{pqr} = M_I^{pqr}$. The conditions for (20) to hold are:

$$\sum_{p,q,r} \frac{\partial J}{\partial M_A^{pqr}} \frac{\partial M_A^{pqr}}{\partial a_{ij}} \equiv 0 \quad ; \quad i,j = 1,2,3. \tag{21}$$

Now, Lie group theory shows [14]: Due to the fact that the parameters a_{ij} build up a group, here $GL(3,\mathbb{R})$, it suffices to demand

$$\sum_{p,q,r} \frac{\partial J}{\partial M^{pqr}} \left.\frac{\partial M_A^{pqr}}{\partial a_{ij}}\right|_{A=I} = 0 \quad ; \quad i,j = 1,2,3. \tag{22}$$

Omitting the proof we only note that functions J which solve (22) will automatically fulfil also eq. (21). This is essentially the only result of Lie group theory we need to make use of here.

We will proceed as follows: We first compute all functions B which exhibit such local invariance with respect to only one parameter $a_{\alpha\beta}$:

$$\sum_{p,q,r} \frac{\partial B}{\partial M^{pqr}} \left.\frac{\partial M_A^{pqr}}{\partial a_{\alpha\beta}}\right|_{A=I} = 0. \tag{23}$$

Next, depending on the solutions B^{pqr} of (23) we compute functions C which are in addition locally invariant with respect to a second group parameter, and so on. We demonstrate this process in some detail for the first parameter and confine ourselves to merely giving the results for the next few steps. Choosing a_{23} as the first parameter to be eliminated we have to solve

$$\sum_{p,q,r} \frac{\partial B}{\partial M^{pqr}} \left.\frac{\partial M_A^{pqr}}{\partial a_{23}}\right|_{A=I} = 0. \tag{24}$$

We obtain from (19)

$$\left.\frac{\partial M_A^{pqr}}{\partial a_{23}}\right|_{A=I} = qM^{p,q-1,r+1} \quad ,$$

so eq. (24) now reads

$$\sum_{p,q,r} \frac{\partial B}{\partial M^{pqr}} qM^{p,q-1,r+1} = 0.$$

This linear partial differential equation of first order is equivalent to a system of ordinary differential equations. Namely, by introducing an independent variable s we obtain

$$(M^{pqr})' = qM^{p,q-1,r+1} \quad , \quad p,q,r = 0,1,2,\ldots$$

where the prime denotes differentiation with respect to s.
We observe that this system splits into disjoint systems involving only moments of equal order $n = p+q+r = p+(q-1)+(r+1)$ which is no surprise since moments of equal order are stable under $GL(3, I\!R)$. In turn, these systems split further by keeping p constant in:

$$\begin{aligned} \left(M^{p,0,n-p}\right)' &= 0 \\ \left(M^{p,1,n-p-1}\right)' &= M^{p,0,n-p} \\ \left(M^{p,2,n-p-2}\right)' &= 2M^{p,1,n-p-1} \\ &\vdots \\ \left(M^{p,n-p,0}\right)' &= (n-p)M^{p,n-p-1,1} \end{aligned}$$

The solution of the equations above is obtained by a simple recursion: Starting with $M^{p,0,n-p} = c^{p,0,n-p}$ we get the formula

$$M^{pqr} = \sum_{\nu=0}^{q} \binom{q}{\nu} c^{p,q-\nu,r+\nu} s^{\nu}$$

which can be easily proved by induction. c^{pqr} denote here constants of integration. For a general group element $\boldsymbol{A}$ we obtain accordingly

$$M_A^{pqr} = \sum_{\nu=0}^{q} \binom{q}{\nu} c_A^{p,q-\nu,r+\nu} s^{\nu}. \tag{25}$$

Now we can in principle eliminate s using any equation (25) linear in s (i.e. $q = 1$):

$$s = \frac{M_A^{p1r}}{M_A^{p,0,r+1}} - \frac{c_A^{p1r}}{c_A^{p,0,r+1}}.$$

However, we obtain a much more systematic parameter elimination process if we resort every time to the lowest possible order. That is especially important taking into account the increased noise vulnerability of moments of higher order [15]. Since moments of order one are occupied by normalizing translation we are forced to consider $p + r = 1$. We choose $p = 0$, $r = 1$ because M_A^{101} could be zero, whereas M_A^{002} not ($M_A^{002} > 0$). Thus, we eliminate s through $s = \frac{M_A^{011}}{M_A^{002}} - \frac{c_A^{011}}{c_A^{002}}$. Inserting this into eq. (25) and separating M' s from c' s we eventually get

$$\sum_{\nu=0}^{q} \binom{q}{\nu} \left(-\frac{M_A^{011}}{M_A^{002}}\right)^{\nu} M_A^{p,q-\nu,r+\nu} = \sum_{\nu=0}^{q} \binom{q}{\nu} \left(-\frac{c_A^{011}}{c_A^{002}}\right)^{\nu} c_A^{p,q-\nu,r+\nu} = \text{const} \tag{26}$$

which can also be easily proved by induction.
Now, we see that the left side of eq. (26) being a solution of (24) taken at a general group

element $\boldsymbol{A}$ is locally invariant with respect to a_{23}. So we define new variables by this very expression:

$$B_A^{pqr} := \sum_{\nu=0}^{q} \binom{q}{\nu} \left(-\frac{M_A^{011}}{M_A^{002}} \right)^{\nu} M_A^{p,q-\nu,r+\nu}. \tag{27}$$

Eq. (27) is at the same time the first step of our group elimination algorithm. It is important to note that by combining moments as in (27) we don't lose control about the dependence of the new variables B_A^{pqr} upon $f(x)$. Inserting (19) into (27) and interchanging the order of summation and integration we get

$$B_A^{pqr} = \int_{\mathbb{R}^3} f(\boldsymbol{x})(\boldsymbol{a}_1^{\mathrm{T}}\boldsymbol{x})^p \sum_{\nu=0}^{q} \binom{q}{\nu} \left(-\frac{M_A^{011}}{M_A^{002}} \right)^{\nu} \left(\frac{\boldsymbol{a}_2^{\mathrm{T}}\boldsymbol{x}}{\boldsymbol{a}_3^{\mathrm{T}}\boldsymbol{x}} \right)^{q-\nu} (\boldsymbol{a}_3^{\mathrm{T}}\boldsymbol{x})^{q+r}\, d\boldsymbol{x}$$

which is easily seen to give

$$B_A^{pqr} = \int_{\mathbb{R}^3} f(\boldsymbol{x})(\boldsymbol{a}_1^{\mathrm{T}}\boldsymbol{x})^p \left[\left(\boldsymbol{a}_2 - \frac{M_A^{011}}{M_A^{002}} \boldsymbol{a}_3 \right)^{\mathrm{T}} \boldsymbol{x} \right]^q (\boldsymbol{a}_3^{\mathrm{T}}\boldsymbol{x})^r\, d\boldsymbol{x}. \tag{28}$$

Thus, the variables B_A^{pqr} are nothing more but moments of $f_{A_1}(x)$: $B_A^{pqr} = M_{A_1}^{pqr}$, where

$$\boldsymbol{A}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{M_A^{011}}{M_A^{002}} \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{A}. \tag{29}$$

Since $M_A^{011} = \boldsymbol{a}_2^{\mathrm{T}}\boldsymbol{a}_3$ and $M_A^{002} = |\boldsymbol{a}_3|^2$ we see that the second and third row of $\boldsymbol{A}_1$ are perpendicular. This indicates eq. (27) to be a first step towards reduction to the orthogonal group. We note further $B_A^{011} \equiv 0$ and

$$\boldsymbol{\Sigma}_{B_A} = \begin{pmatrix} B^{200} & B^{110} & B^{101} \\ B^{110} & B^{020} & B^{011} \\ B^{101} & B^{011} & B^{002} \end{pmatrix}_A = \begin{pmatrix} M_A^{200} & V/M_A^{002} & M_A^{101} \\ V/M_A^{002} & U/M_A^{002} & 0 \\ M_A^{101} & 0 & M_A^{002} \end{pmatrix}.$$

Here we denote by U, V and the later needed W the following minors in $\boldsymbol{\Sigma}_{M_A} = \boldsymbol{A}\boldsymbol{A}^{\mathrm{T}}$.

$$U := \begin{vmatrix} M_A^{020} & M_A^{011} \\ M_A^{011} & M_A^{002} \end{vmatrix}, \quad V := \begin{vmatrix} M_A^{110} & M_A^{101} \\ M_A^{011} & M_A^{002} \end{vmatrix}, \quad W := \begin{vmatrix} M_A^{110} & M_A^{101} \\ M_A^{020} & M_A^{011} \end{vmatrix}.$$

Now, for the derivation of the second step of our algorithm we make use of eq. (28) and repeat the outlined process for a second parameter a_{13}. We are led to the variables

$$C_A^{pqr} := \sum_{\nu=0}^{p} \binom{p}{\nu} \left(-\frac{B_A^{101}}{B_A^{002}} \right)^{\nu} B_A^{p-\nu,q,r+\nu}, \tag{30}$$

which relate to $f(\boldsymbol{x})$ as

$$C_A^{pqr} = \int_{\mathbb{R}^3} f(\boldsymbol{x}) \left[\left(\boldsymbol{a}_1 - \frac{M_A^{101}}{M_A^{002}} \boldsymbol{a}_3 \right)^{\mathrm{T}} \boldsymbol{x} \right]^p \left[\left(\boldsymbol{a}_2 - \frac{M_A^{011}}{M_A^{002}} \boldsymbol{a}_3 \right)^{\mathrm{T}} \boldsymbol{x} \right]^q (\boldsymbol{a}_3^{\mathrm{T}} \boldsymbol{x})^r \, d\boldsymbol{x}$$

and have the property $C_A^{pqr} = M_{A_2}^{pqr}$. In $\boldsymbol{A}_2$, second *and* first row are perpendicular to the third row.

Omitting further details we give the next steps which are obtained by eliminating in the same manner the parameters a_{12}, a_{11}, a_{22} and a_{33}:

$$D_A^{pqr} := \sum_{\nu=0}^{p} \binom{p}{\nu} \left(-\frac{C_A^{110}}{C_A^{020}} \right)^{\nu} C_A^{p-\nu, q+\nu, r}, \tag{31}$$

$$E_A^{pqr} := \frac{D_A^{pqr}}{\left(D_A^{200}\right)^{p/2} \left(D_A^{020}\right)^{q/2} \left(D_A^{002}\right)^{r/2}}. \tag{32}$$

Like all previous intermediate variables, E_A^{pqr} are moments as well: $E_A^{pqr} = M_R^{pqr}$; and since $\boldsymbol{\Sigma}_{E_A} = \boldsymbol{R}\boldsymbol{R}^{\mathrm{T}} = \boldsymbol{I}$ we see that we have achieved a reduction to the orthogonal group. Examining $\boldsymbol{R}$ a little closer we find

$$\boldsymbol{R} = \begin{pmatrix} \boldsymbol{r}_1^{\mathrm{T}} \\ \boldsymbol{r}_2^{\mathrm{T}} \\ \boldsymbol{r}_3^{\mathrm{T}} \end{pmatrix} \quad \text{with}$$

$$\boldsymbol{r}_1 = \frac{U\boldsymbol{a}_1 - V\boldsymbol{a}_2 + W\boldsymbol{a}_3}{|\boldsymbol{A}|\sqrt{U}}, \quad \boldsymbol{r}_2 = \frac{M_A^{002}\boldsymbol{a}_2 - M_A^{011}\boldsymbol{a}_3}{\sqrt{M_A^{002}U}}, \quad \boldsymbol{r}_3 = \frac{\boldsymbol{a}_3}{\sqrt{M_A^{002}}}.$$

A simple calculation shows:

$$\det(\boldsymbol{R}) = \boldsymbol{r}_1^{\mathrm{T}}(\boldsymbol{r}_2 \times \boldsymbol{r}_3) = \frac{\det(\boldsymbol{A})}{|\boldsymbol{A}|} = \operatorname{sgn}(\det(\boldsymbol{A})) =: \epsilon_A$$

We observe that the information about the sign of $\det(\boldsymbol{A})$ (mirror images) has been preserved in the "handedness" of the matrix $\boldsymbol{R}$. Although ϵ_A is not yet known we define variables

$$\hat{E}_A^{pqr} := \epsilon_A^p E_A^{pqr} = M_P^{pqr} \quad \text{with} \tag{33}$$

$$\boldsymbol{P} = \begin{pmatrix} \epsilon_A & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{R} \quad \text{and} \quad \boldsymbol{P} \in SO(3). \tag{34}$$

We will be using $\hat{E}_A^{pqr}$ as if ϵ_A were known and return to this point at the end of our computations to see how we have to modify here the resulting algorithm.

5.4 Invariants of the group $SO(3)$

Our starting point are now the moments

$$\hat{E}_A^{pqr} = M_P^{pqr} = \int_{\mathbb{R}^3} f_P(\boldsymbol{x}) x^p y^q z^r \, d\boldsymbol{x} \quad \text{with} \quad \boldsymbol{P} \in SO(3). \tag{35}$$

We choose the parameterization $\boldsymbol{P}(\alpha,\beta,\gamma) = \boldsymbol{P}_z(\gamma)\boldsymbol{P}_y(\beta)\boldsymbol{P}_x(\alpha)$ with $\boldsymbol{P}_{x,y,z}(\omega)$ denoting rotation around axis x,y or z respectively by an angle of ω. Note that this parameterization is unique for $0 \leq \alpha, \gamma < 2\pi$ and $-\pi/2 \leq \beta \leq \pi/2$. The infinitesimal action of $\boldsymbol{P}$ in $\mathbb{R}^3$ is:

$$\left.\frac{\partial(\boldsymbol{P}\boldsymbol{x})}{\partial\alpha}\right|_I = \begin{pmatrix} 0 \\ -z \\ y \end{pmatrix}, \quad \left.\frac{\partial(\boldsymbol{P}\boldsymbol{x})}{\partial\beta}\right|_I = \begin{pmatrix} z \\ 0 \\ -x \end{pmatrix}, \quad \left.\frac{\partial(\boldsymbol{P}\boldsymbol{x})}{\partial\gamma}\right|_I = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}. \tag{36}$$

Now, we can try to continue applying the infinitesimal method directly to the variables $\hat{E}_A^{pqr}$. The subspace

$$Q_n := < x^n, x^{n-1}y, \ldots, x^p y^q z^r, \ldots, z^n >$$

with dimension $\frac{(n+1)(n+2)}{2}$ spanned by all monomials of order n was invariant under $GL(3,\mathbb{R})$. So it is a fortiori invariant under $SO(3) \subset GL(3,\mathbb{R})$ and we are left again with finite dimensional problems. However, the differential equations emerging by trying to eliminate successively the parameters α, β and γ in Q_n are getting more and more complex and we did not succeed in solving them directly. But we can do better: It is well known that Q_n splits under $SO(3)$ further in smaller invariant subspaces. Q_n is now reducible. We call the irreducible invariant subspaces of Q_n under $SO(3)$ by M_{nl}. They have been extensively studied in the past in conjunction with quantum mechanics [16] and more recently with image understanding and processing ([17], [18]) so we outline here only some needed basic results for easy reference:

Since $SO(3)$ is compact Q_n is decomposed by a direct sum :

$$Q_n = M_{nn} \oplus M_{n,n-2} \oplus M_{n,n-4} \oplus \cdots \oplus \begin{cases} M_{n1} \text{ if } n \text{ odd} \\ M_{n0} \text{ if } n \text{ even} \end{cases}$$

$$M_{nl} = (x^2 + y^2 + z^2)^{(n-l)/2} \cdot K_l \,,$$

K_l being the space of all homogeneous polynomials of order l in three cartesian coordinates which are eigenfunctions of the Laplacian operator (harmonic polynomials). The dimension of K_l and M_{nl} is $(2l+1)$ and we verify:

$$\begin{aligned} \dim(M_{nn}) + \dim(M_{n,n-2}) + \cdots + \dim(M_{n1} \text{ or } M_{n0}) &= \\ = (2n+1) + (2n-3) + (2n-7) + \cdots + (3 \text{ or } 1) &= \\ = \frac{(n+1)(n+2)}{2} \ (\text{ in both cases! }) &= \dim(Q_n). \end{aligned}$$

We can represent basis functions $\varepsilon_{nl}^{m}(\boldsymbol{x})$ spanning M_{nl} as follows:

$$\varepsilon_{nl}^{m}(\boldsymbol{x}) = (x^2+y^2+z^2)^{\frac{n-l}{2}} \left[\frac{j}{2}(x+jy)\right]^m z^{l-m} \sum_{\nu=0}^{\lfloor\frac{l-m}{2}\rfloor} \left(-\frac{1}{4}\right)^{\nu} \binom{l}{\nu} \binom{l-\nu}{m+\nu} \left(\frac{x^2+y^2}{z^2}\right)^{\nu} \tag{37}$$

if $0 \leq m \leq l$

$$\varepsilon_{nl}^{-m}(\boldsymbol{x}) = (-1)^m \cdot \varepsilon_{nl}^{m}(\boldsymbol{x})^*$$
$$\varepsilon_{nl}^{m}(\boldsymbol{x}) \equiv 0 \text{ if } |m| > l$$

where the asterisc denotes complex conjugation. These are spherical harmonic functions on the sphere with radius $(x^2+y^2+z^2)^{n/2}$ (we omit the proof).

Since $(n-l)$ is even and $(l-m) \geq 2 \cdot \lfloor\frac{l-m}{2}\rfloor$, $\varepsilon_{nl}^{m}(\boldsymbol{x})$ are homogeneous polynomials in x, y and z of order n. The invariance of M_{nl} under $SO(3)$ is reflected also by the infinitesimal changes of $\varepsilon_{nl}^{m}(\boldsymbol{x})$ with respect to the parameters α, β and γ of $SO(3)$. Using (36) and (37) we can show:

$$\left.\frac{\partial}{\partial\alpha}\varepsilon_{nl}^{m}(\boldsymbol{Px})\right|_I = \frac{l-m+1}{2} \cdot \varepsilon_{nl}^{m-1}(\boldsymbol{x}) - \frac{l+m+1}{2} \cdot \varepsilon_{nl}^{m+1}(\boldsymbol{x}) \tag{38}$$

$$\left.\frac{\partial}{\partial\beta}\varepsilon_{nl}^{m}(\boldsymbol{Px})\right|_I = j\left(\frac{l-m+1}{2} \cdot \varepsilon_{nl}^{m-1}(\boldsymbol{x}) + \frac{l+m+1}{2} \cdot \varepsilon_{nl}^{m+1}(\boldsymbol{x})\right) \tag{39}$$

$$\left.\frac{\partial}{\partial\gamma}\varepsilon_{nl}^{m}(\boldsymbol{Px})\right|_I = jm\varepsilon_{nl}^{m}(\boldsymbol{x}). \tag{40}$$

This is a result also known from analyzing the Lie algebra of $SO(3)$ [19].

Prior to continue applying the Lie method in order to eliminate the remaining three parameters α, β and γ of the group $SO(3)$, we now compute expansion coefficients of $f_P(\boldsymbol{x})$ with respect to the system of basis functions $\varepsilon_{nl}^{m}(\boldsymbol{x})$, rather than to that of the monomials $x^p y^q z^r$.

$$(F_A)_{nl}^{m} := \int_{\mathbb{R}^3} f_P(\boldsymbol{x})\varepsilon_{nl}^{m}(\boldsymbol{x})^* \, d\boldsymbol{x}. \tag{41}$$

It is important to note that we have not to compute the integral above explicitly. Thanks to eq. (37) and to the fact that $\varepsilon_{nl}^{m}(\boldsymbol{x})$ are homogeneous polynomials in x, y and z we can express the $(F_A)_{nl}^{m}$' s as linear combinations of the $\hat{E}_A^{pqr}$' s of equal order $n = p+q+r$. To illustrate this point a little we display a few examples of low order which can be verified using eq. (41), (37) and (35).

$n = 3$:

$$(F_A)_{33}^{3} = (1/8)[(3\hat{E}_A^{210} - \hat{E}_A^{030}) + j(\hat{E}_A^{300} - 3\hat{E}_A^{120})]$$

$$(F_A)_{33}^{2} = -(3/4)[(\hat{E}_A^{201} - \hat{E}_A^{021}) - j \cdot 2\hat{E}_A^{111}]$$

$$\vdots$$

$$
\begin{aligned}
(F_A)_{31}^{1} &= -(1/2)[(\hat{E}_A^{210} + \hat{E}_A^{030} + \hat{E}_A^{012}) + j(\hat{E}_A^{300} + \hat{E}_A^{120} + \hat{E}_A^{102})] \\
(F_A)_{31}^{0} &= \hat{E}_A^{201} + \hat{E}_A^{021} + \hat{E}_A^{003}
\end{aligned}
$$

$n = 4$:

$$
\begin{aligned}
(F_A)_{44}^{4} &= (1/16)[(\hat{E}_A^{400} - 6\hat{E}_A^{220} + \hat{E}_A^{040}) - j \cdot 4(\hat{E}_A^{310} - \hat{E}_A^{130})] \\
&\vdots \\
(F_A)_{40}^{0} &= \hat{E}_A^{400} + \hat{E}_A^{040} + \hat{E}_A^{004} + 2(\hat{E}_A^{220} + \hat{E}_A^{202} + \hat{E}_A^{022}).
\end{aligned}
$$

We also note

$$
(F_A)_{nl}^{-m} = (-1)^m \cdot \left((F_A)_{nl}^{m}\right)^* \quad \text{and} \quad (F_A)_{nl}^{m} \equiv 0 \quad \text{if} \quad |m| > l.
$$

Equations (38), (39) and (40) are transferred directly to the coefficients $(F_A)_{nl}^{m}$:

$$
\frac{\partial}{\partial\alpha}(F_A)_{nl}^{m}\Big|_I = \frac{l-m+1}{2} \cdot F_{nl}^{m-1} - \frac{l+m+1}{2} \cdot F_{nl}^{m+1} \tag{42}
$$

$$
\frac{\partial}{\partial\beta}(F_A)_{nl}^{m}\Big|_I = -j\left(\frac{l-m+1}{2} \cdot F_{nl}^{m-1} + \frac{l+m+1}{2} \cdot F_{nl}^{m+1}\right) \tag{43}
$$

$$
\frac{\partial}{\partial\gamma}(F_A)_{nl}^{m}\Big|_I = -jmF_{nl}^{m}. \tag{44}
$$

We can use the phase angle of any coefficient with $|m| = 1$ to build γ-invariants. However, following a similar reasoning as in page 16 after eq. (25) we choose $(F_A)_{31}^{1}$. With the notation $(F_A)_{nl}^{m} =: \left|(F_A)_{nl}^{m}\right| e^{j(\phi_A)_{nl}^{m}}$ we immediately check that variables

$$
(G_A)_{nl}^{m} := (F_A)_{nl}^{m} \cdot e^{-jm(\phi_A)_{31}^{1}} \tag{45}
$$

are locally invariant with respect to γ. Of course, the definition above doesn't work if $(F_A)_{31}^{1} = 0$. In this rather unlikely case we must resort to some other coefficient with $|m| = 1$.

Now we could proceed as in section 5.3., i.e. first compute the relationship between $(G_A)_{nl}^{m}$ and $f(\boldsymbol{x})$ and then derive formulas for the elimination of α and β. However, that would demand explicitly invoking the irreducible representations of $SO(3)$. We would like to avoid this here by computing the infinitesimal changes of $(G_A)_{nl}^{m}$ directly from eq. (42), (43) and (45) and express the result in terms of the variables $(G_A)_{nl}^{m}$ themselves. A long but elementary calculation gives:

$$\begin{aligned}
\frac{\partial}{\partial\alpha}(G_A)^m_{nl}\Big|_I &= \left(m\frac{G^0_{31}}{4G^1_{31}}\cdot G^m_{nl} - \frac{l+m+1}{2}\cdot G^{m+1}_{nl}\right)e^{j\phi^1_{31}} - \\
&- \left(m\frac{G^0_{31}}{4G^1_{31}}\cdot G^m_{nl} - \frac{l-m+1}{2}\cdot G^{m-1}_{nl}\right)e^{-j\phi^1_{31}} \\
\frac{\partial}{\partial\beta}(G_A)^m_{nl}\Big|_I &= j\left[\left(m\frac{G^0_{31}}{4G^1_{31}}\cdot G^m_{nl} - \frac{l+m+1}{2}\cdot G^{m+1}_{nl}\right)e^{j\phi^1_{31}} + \right. \\
&+ \left.\left(m\frac{G^0_{31}}{4G^1_{31}}\cdot G^m_{nl} - \frac{l-m+1}{2}\cdot G^{m-1}_{nl}\right)e^{-j\phi^1_{31}}\right].
\end{aligned}$$

Unfortunately, it is not possible here to get rid of the coefficients $e^{j\phi^1_{31}}$ which belong to the previous system of variables. However, we succeeded in solving the system above in two steps: Firstly, we compute solutions of

$$(G^m_{nl})' = m\frac{G^0_{31}}{4G^1_{31}}\cdot G^m_{nl} - \frac{l+m+1}{2}\cdot G^{m+1}_{nl}$$

where the prime denotes again differentiation with respect to an artificially introduced independent variable s. We solve this system in the manner outlined in section 5.3. for a simpler case there. Omitting the details we present the solution in the form

$$\sum_{\nu=m}^{l}\binom{l+\nu}{\nu-m}\left(-\frac{G^0_{31}}{2}\right)^{\nu-m}\cdot\frac{G^\nu_{nl}}{(G^1_{31})^\nu} = \text{const}$$

and use the left side of this equation for the definition of new variables $(H_A)^m_{nl}$. However, we observe from eq. (45) that

$$(G_A)^0_{31} = (F_A)^0_{31} \quad\text{and}\quad \frac{(G_A)^\nu_{nl}}{\left[(G_A)^1_{31}\right]^\nu} = \frac{(F_A)^\nu_{nl}}{\left[(F_A)^1_{31}\right]^\nu}.$$

Therefore we can ignore (45) and define instead directly after eq. (41) of our algorithm:

$$(H_A)^m_{nl} := \sum_{\nu=m}^{l}\binom{l+\nu}{\nu-m}\left(-\frac{(F_A)^0_{31}}{2}\right)^{\nu-m}\cdot\frac{(F_A)^\nu_{nl}}{\left[(F_A)^1_{31}\right]^\nu}. \tag{46}$$

Examination of these variables for $n=3,\ l=1$ reveals $(H_A)^1_{31}\equiv 1$, $(H_A)^0_{31}\equiv 0$ and

$$(H_A)^{-1}_{31} = -\left[\left(\frac{(F_A)^0_{31}}{2}\right)^2 + \left|(F_A)^1_{31}\right|^2\right] = H^{-1}_{31} =: -C^2. \tag{47}$$

$(H_A)^1_{31}$ and $(H_A)^0_{31}$ are trivial invariants (equal for all objects) and indicate the degrees of freedom sacrificed by executing eq.(46) of our algorithm. On the other hand H^{-1}_{31} turns out to be the first true invariant of $GL(3,I\!R)$. We simply check

$$\frac{\partial C}{\partial \alpha} \equiv 0 \quad \frac{\partial C}{\partial \beta} \equiv 0 \quad \frac{\partial C}{\partial \gamma} \equiv 0.$$

It remains to investigate the behaviour of the variables $(H_A)^m_{nl}$ under changes of α and β. Using (46), (42) and (43) we obtain

$$\left.\frac{\partial}{\partial \alpha}(H_A)^m_{nl}\right|_I = \frac{1}{F^1_{31}}\left(\frac{l+m+1}{2}C^2 \cdot H^{m+1}_{nl} + \frac{l-m+1}{2}H^{m-1}_{nl}\right) \tag{48}$$

$$\left.\frac{\partial}{\partial \beta}(H_A)^m_{nl}\right|_I = -\frac{j}{F^1_{31}}\left(\frac{l+m+1}{2}C^2 \cdot H^{m+1}_{nl} + \frac{l-m+1}{2}H^{m-1}_{nl}\right). \tag{49}$$

Because the overall common factors in eq. (48) and (49) don't affect the derivation of invariant functions depending on $(H_A)^m_{nl}$ we see that we have to solve

$$(H^m_{nl})' = \frac{l+m+1}{2}C^2 \cdot H^{m+1}_{nl} + \frac{l-m+1}{2}H^{m-1}_{nl}$$

in *both* cases. Thus, denoting by

$$\boldsymbol{H}_{nl} \text{ the vector } \begin{pmatrix} H^l_{nl} \\ H^{l-1}_{nl} \\ \vdots \\ H^{-l}_{nl} \end{pmatrix} \text{ and by } \boldsymbol{A}_l \text{ the matrix } \begin{pmatrix} 0 & 1/2 & & & \\ lC^2 & 0 & 1 & \mathbf{0} & \\ & (l-1/2)C^2 & 0 & \ddots & \\ & & \ddots & \ddots & l \\ & \mathbf{0} & & 1/2C^2 & 0 \end{pmatrix}$$

we are faced with the linear problem $(\boldsymbol{H}_{nl})' = \boldsymbol{A}_l \cdot \boldsymbol{H}_{nl}$. This equation can be diagonalized by a transform matrix $\boldsymbol{T}_l = \text{diag}(1, C, C^2, \ldots, C^{2l}) \cdot \mathcal{T}_l$ where $\mathcal{T}_l = (\tau^m_{l\lambda})$ and the matrix elements $\tau^m_{l\lambda}$ are given by the formula

$$\tau^m_{l\lambda} = [\text{sgn}(m)]^{l-\lambda}[\text{sgn}(\lambda)]^{l-m}(-1)^l \sum_{k=\max(|\lambda|,|m|)}^{l} (-1)^k \binom{l-|\lambda|}{k-|\lambda|}\binom{l+|\lambda|}{k-|m|}. \tag{50}$$

The sgn-function is here defined by

$$\text{sgn}(n) = \begin{cases} 1 & \text{if } n \geq 0 \\ -1 & \text{if } n < 0 \end{cases}$$

slightly departing from the usual definition. Furthermore, it can be shown that

$$\boldsymbol{T}_l^{-1} = 2^{-2l}\text{diag}(1, C^{-1}, C^{-2}, \ldots, C^{-2l})\boldsymbol{T}_l\text{diag}(1, C^{-1}, C^{-2}, \ldots, C^{-2l})$$

and we obtain the eigenvalues of $\boldsymbol{A}_l$ as entries of the diagonal matrix

$$\boldsymbol{T}_l^{-1}\boldsymbol{A}_l\boldsymbol{T}_l = \mathrm{diag}[lC, (l-1)C, \ldots, -lC].$$

From the results above we see that the next step of the algorithm must be the definition of new variables $(I_A)_{nl}^m$ by

$$(\boldsymbol{I}_A)_{nl} = \begin{pmatrix} (I_A)_{nl}^l \\ \vdots \\ (I_A)_{nl}^{-l} \end{pmatrix} := \boldsymbol{T}_l^{-1}(\boldsymbol{H}_A)_{nl}. \tag{51}$$

Because now due to diagonalization $(I_{nl}^m)' = mCI_{nl}^m$ we can normalize these variables in a last step by choosing again some particular variable with $|m| = 1$ and use its real or imaginary part:

$$(J_A)_{nl}^m = \frac{(I_A)_{nl}^m}{\left[\mathrm{Re}\{(I_A)_{33}^1\}\right]^m} \tag{52}$$

These are full invariants of the group $GL(3, I\!\!R)$ up to ϵ_A. Now returning to eq. (33) we see that if we use instead of the *unknown* coefficients $\hat{E}_A^{pqr}$ the *known* ones E_A^{pqr} we can compute the effect of ϵ_A on our invariants. It is easy to see that

$$J_{nl}^m(\epsilon_A) = \mathrm{Re}\{J_{nl}^m\} + j\epsilon_A \mathrm{Im}\{J_{nl}^m\}.$$

If we want to discriminate between mirror objects, we can use the invariants J_{nl}^m directly. Otherwise, we build

$$K_{nl}^m := \mathrm{Re}\{J_{nl}^m\} + j\mathrm{sgn}(\mathrm{Im}\{J_{33}^1\}) \cdot \mathrm{Im}\{J_{nl}^m\}. \tag{53}$$

Note that defining K_{nl}^m as $\mathrm{Re}\{J_{nl}^m\} + j|\mathrm{Im}\{J_{nl}^m\}|$ would introduce ambiguities in the invariants. The set $\{K_{nl}^m\}$ is a complete set of invariants for the full group $GL(3, I\!\!R)$.

5.5 Discussion

In conclusion we first indicate the equations which have to be really executed by a digital machine:
a) A digitized form of eq. (18) using some fast algorithm.
b) Equations (27), (30), (31) and (32) will perform the reduction to the orthogonal group.
c) Computation of variables $(F_A)_{nl}^m$ combining the E_A^{pqr} 's linearly as equations (41), (37) and (35) indicate.
d) Equations (46), (51), (52) and if also invariance with respect to reflections is desired eq. (53).

We observe that all steps of our algorithm combine the variables of the preceding step *linearly* with the exception of using one or two members of low order out of them nonlinearly for normalization. Every time this occurs one degree of freedom is sacrificed but not more. The reduction to the orthogonal group occurs by eliminating six parameters out of nine of the full group $GL(3, \mathbb{R})$. Accordingly six moments (all moments of order 2) are set to fixed values. Equations (41) and (51) introduce only a coordinate transformation in the function space. Eq. (46) sacrifices two degrees of freedom (double step) and eq. (52) one ($\mathrm{Re}\{(J_A)^1_{33}\} = 1$) and finally eq. (53) normalizes to $\mathrm{Im}\{K^1_{33}\} > 0$.

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