Analytic Reduction of the Kruppa Equations

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Abstract. Given the fundamental matrix between a pair of images taken by a nonstationary projective camera with constant internal parameters, we show how to use the two independent Kruppa equations in order to explicitly cut down the number of parameters of the Kruppa matrix $\mathbf{KK}^T$ by exactly two. Thus, we derive a procedure which results in a closed formula for the Kruppa matrix that depends on exactly three remaining parameters. This formula allows an easy incorporation of the positivity constraint and admits of an interpretation in terms of the image of the horopter. We focus on the general case where the camera motion is unknown and not restricted to some special type. Solutions of the Kruppa equations given three fundamental matrices have been attempted in the past by iterative numerical methods that are searching in multidimensional spaces. As an application of the reduced Kruppa matrix mentioned above we also outline how this problem can be analytically reduced to the determination of the real, positive roots of a polynomial of 14-th degree in one variable.

1 Introduction

The possibility of calibrating a moving projective camera on-the-fly has been indicated about ten years ago in the seminal papers [2] and [11]. The resulted technique referred to as self-calibration greatly increases the flexibility in using a camera for computer vision purposes because there is no need any more to tediously precalibrate the camera using special calibration objects. Self-calibration is performed only on the basis of image measurements and despite unknown camera motion. The basic observation was that there is a relation connecting the fundamental matrix between two views and the calibration matrix of the camera which contains the internal parameters. This relation is given in terms of the Kruppa equations [6] dating back to 1913. Since the fundamental matrix can be computed from image measurements, this gives conditions for the calibration matrix from image measurements alone. However, barring special restricted cases, the solution of the Kruppa equations turned out to be extremely difficult due to the nonlinear nature of the problem. Although there have been designed numerical methods ([9], [16]) for their solution, many problems like high computational cost motivated researchers to look for alternatives of self-calibration.

These efforts produced several techniques like the formulation in terms of the absolute quadric [15], combinations of scene and auto-calibration constraints [7], or stratified approaches by first computing the plane at infinity [12].

In this paper we investigate in some detail the space of the solutions of the Kruppa equations between two views and derive an intricate relation with the image of the horopter. According to this relation, there is a one dimensional infinity of well specified point pairs on the image of the horopter, each generating, in a sense to be explained below, a different three-dimensional subspace of the six-dimensional space of symmetrical 3×3 matrices. We show that every point on any of those subspaces solves the Kruppa equations and that conversely any solution of the Kruppa equations must lie in some of those subspaces. The parameter of the point pair above and the projective position within the 3D subspace it generates, are the new three parameters of the Kruppa matrix which can even be given in closed form and in dependence of these three parameters. The incorporation of the positivity constraint is easily accomplished.

As an application of the derived parameterisation of the reduced Kruppa matrix we show how to use it in order to reduce the six Kruppa equations, given three fundamental matrices, to a single polynomial equation of 14-th degree in one variable.

2 Self-calibration from the infinity homography

Considering two projective cameras $\mathbf{P}_I$ and $\mathbf{P}_J$ with the same calibration matrix $\mathbf{K}$ we can choose the world coordinate system aligned with the camera $\mathbf{P}_I$, such that the two cameras described by homogeneous 3×4 matrices will read: $\mathbf{P}_I \sim \mathbf{K}(\mathbf{I}; \mathbf{0})$ and $\mathbf{P}_J \sim \mathbf{K}\mathbf{R}_J(\mathbf{I}; -\mathbf{c}_J)$ where $\mathbf{K}$ denotes the common upper right triangular matrix that contains the internal parameters of the cameras, $\mathbf{R}_J$ denotes the 3×3 rotation matrix describing the orientation of the camera $\mathbf{P}_J$ with respect to $\mathbf{P}_I$, and $\mathbf{c}_J$ is the camera center of the camera $\mathbf{P}_J$. Of course, $\mathbf{P}_I$ and $\mathbf{P}_J$ may and in general will denote two instances of the same camera in two different positions. It is easy to see that between the images of points at infinity will lie the so called infinity homography $\mathbf{H}^\infty_{JI} = \mathbf{K}\mathbf{R}_J\mathbf{K}^{-1}$ with determinant normalised to one. Now it is also easy to eliminate the unknown rotation $\mathbf{R}_J$ from this equation giving

$$\mathbf{H}^\infty_{JI}\mathbf{K}\mathbf{K}^T\mathbf{H}^{\infty T}_{JI} = \mathbf{K}\mathbf{K}^T \ . \tag{1}$$

The positive definite symmetric matrix $\mathbf{K}\mathbf{K}^T$ also called Kruppa matrix [1] contains the same information as the calibration matrix $\mathbf{K}$ since we can derive $\mathbf{K}$ from $\mathbf{K}\mathbf{K}^T$ by a Cholesky factorisation. The geometrical meaning of the Kruppa matrix is very well known to be the description of the dual image of the absolute conic. The algebraic structure of equation (1) is that of an eigenvector one: The 6-dimensional vector containing all elements in the upper right half of $\mathbf{K}\mathbf{K}^T$ is an eigenvector of a certain 6×6 matrix $\mathbf{H}^{\infty[2]}_{JI}$, depending homomorphically on $\mathbf{H}^\infty_{JI}$, to the eigenvalue 1. However, since the eigenvalues of $\mathbf{H}^\infty_{JI}$ are the same as the eigenvalues of $\mathbf{R}_J$ they are given by the set $\{e^{j\phi}, e^{-j\phi}, 1\}$ where ϕ denotes the angle of rotation of $\mathbf{R}_J$. In turn, the eigenvalues of $\mathbf{H}_{JI}^{\infty[2]}$ will be all six products of the three eigenvalues above taken two at a time (including repetitions) which gives the set $\{e^{j2\phi}, 1, e^{j\phi}, e^{-j2\phi}, e^{-j\phi}, 1\}$. We observe that $\mathbf{H}_{JI}^{\infty[2]}$ will possess a double eigenvalue 1 which means that $\mathbf{KK}^T$ has to be sought in a two dimensional eigenspace. Indeed, with $\mathbf{a}_J$ denoting the axis of rotation of $\mathbf{R}_J$ it is well known that we also will have $\mathbf{H}_{JI}^{\infty}\mathbf{K}\mathbf{a}_J = \mathbf{K}\mathbf{R}_J\mathbf{a}_J = \mathbf{K}\mathbf{a}_J$ and consequently also $\mathbf{H}_{JI}^{\infty}\mathbf{K}\mathbf{a}_J\mathbf{a}_J^T\mathbf{K}^T\mathbf{H}_{JI}^{\infty T} = \mathbf{K}\mathbf{a}_J\mathbf{a}_J^T\mathbf{K}^T$. Thus, any linear combination of $\mathbf{KK}^T$ and $\mathbf{K}\mathbf{a}_J\mathbf{a}_J^T\mathbf{K}^T$ will solve equation (1). Therefore, a single image pair won't be enough for self-calibration from the infinity homography in the general case. To resolve the ambiguity, a second image pair and the associated infinity homography is necessary.

Obtaining the infinity homography or the plane at infinity is known to be usually very hard to achieve if no special scene structure or restricted camera motion like for example a stationary rotating camera [3] can be assumed. In contrast, the fundamental matrix is far easier to obtain but the resulting Kruppa equations are considerably weaker allowing therefore a richer solution space that complicates the problem. In the next section we formulate the Kruppa equations and start with the investigation of their solution space for one image pair.

3 The Kruppa equations

Algebraically, the Kruppa equations are immediately obtained from equation (1). Denoting with $\mathbf{e}_{JI}$ the epipole on image J and with $[\mathbf{e}_{JI}]_\times$ the 3×3 rank two skew symmetrical matrix that computes the cross product with $\mathbf{e}_{JI}$ and multiplying (1) from both sides with $[\mathbf{e}_{JI}]_\times$ one gets $[\mathbf{e}_{JI}]_\times\mathbf{H}_{JI}^{\infty}\mathbf{KK}^T\mathbf{H}_{JI}^{\infty T}[\mathbf{e}_{JI}]_\times \sim [\mathbf{e}_{JI}]_\times\mathbf{KK}^T[\mathbf{e}_{JI}]_\times$. Since the product $[\mathbf{e}_{JI}]_\times\mathbf{H}_{JI}^{\infty}$ is recognised to be the fundamental matrix $\mathbf{F}_{JI}$ assigning to points on image I epipolar lines on image J, the derived Kruppa equations read in matrix form

$$\mathbf{F}_{JI}\mathbf{KK}^T\mathbf{F}_{JI}^T \sim [\mathbf{e}_{JI}]_\times\mathbf{KK}^T[\mathbf{e}_{JI}]_\times \ . \tag{2}$$

Note that since the fundamental matrix as well as the matrix $[\mathbf{e}_{JI}]_\times$ is singular, we cannot normalise with respect to the determinant any more.

It is well known that the Kruppa equations for a single image pair are equivalent to only two independent scalar equations. This fact can be shown in a number of ways as for example using the singular value decomposition of the fundamental matrix ([4], [8]). Initially, we will follow a different approach starting from the analysis of the solution space of equation (1) given in the previous section.

We first mention the well known observation that any solution of equations (1) will a fortiori also solve the Kruppa equations (2). But the space of possible solutions $\mathbf{X}$ of the Kruppa equations $\mathbf{F}_{JI}\mathbf{X}\mathbf{F}_{JI}^T \sim [\mathbf{e}_{JI}]_\times\mathbf{X}[\mathbf{e}_{JI}]_\times$, where $\mathbf{X}$ is a symmetric 3×3 matrix, is now much richer, mainly due to the singularity of $\mathbf{F}_{JI}$ and $[\mathbf{e}_{JI}]_\times$. Consider $\mathbf{X}_0 \sim \mathbf{e}_{JI}\mathbf{e}_{IJ}^T + \mathbf{e}_{IJ}\mathbf{e}_{JI}^T$. It is easy to see that $\mathbf{F}_{JI}\mathbf{X}_0\mathbf{F}_{JI}^T = \mathbf{0} = [\mathbf{e}_{JI}]_\times\mathbf{X}_0[\mathbf{e}_{JI}]_\times$. Since adding an arbitrary multiple of $\mathbf{X}_0$ to any solution $\mathbf{X}$ of (2) would only add the zero matrix to both sides of the equation, the sum $\mathbf{X} + \lambda \mathbf{X}_0$ would be a solution of (2) as well for all λ. The linear subspace $< \mathbf{KK}^T, \mathbf{Ka}_J \mathbf{a}_J^T \mathbf{K}^T, \mathbf{X}_0 >$ spanned by these three symmetric matrices is therefore already a 3-dimensional linear solutions-subspace within the six-dimensional space of symmetric 3×3 matrices that contains the sought solution $\mathbf{KK}^T$. We will show: There is an infinity of such 3-dimensional linear subspaces, all points $\mathbf{X}$ of which solve $\mathbf{F}_{JI} \mathbf{X} \mathbf{F}_{JI}^T \sim [\mathbf{e}_{JI}]_\times \mathbf{X} [\mathbf{e}_{JI}]_\times$. Each one of these subspaces is generated by a point pair on the image of the horopter.

For the further description we need some basic and well known facts concerning the horopter and its image (cf. [5]) as well as a suitably normalised canonical parametric equation [13] of the latter which we introduce in the subsection 4.1.

4 The horopter and its image

The horopter is defined as a curve in space, all points of which map onto identical positions on the image planes I and J, i.e. we have $\mathbf{P}_I \mathbf{X} \sim \mathbf{P}_J \mathbf{X} \sim \mathbf{x}$ if and only if the space point $\mathbf{X}$ is on the horopter. Since the image $\mathbf{x}$ is self corresponding it must fulfil the epipolar condition $\mathbf{x}^T \mathbf{F}_{JI} \mathbf{x} = 0$. This shows that with $\mathbf{F}_{JI}^s$ being the symmetric part of the fundamental matrix, $\mathbf{F}_{JI}^s \sim \mathbf{F}_{JI} + \mathbf{F}_{JI}^T$, we will have $\mathbf{x}^T \mathbf{F}_{JI}^s \mathbf{x} = 0$ which means that the image point $\mathbf{x}$ must lie on the conic that is described by the symmetric and generally regular matrix $\mathbf{F}_{JI}^s$. Consequently, this conic is the image of the horopter. One easily verifies that both epipoles $\mathbf{e}_{IJ}$ and $\mathbf{e}_{JI}$ are lying on the image of the horopter. Furthermore, even the antisymmetric part $\mathbf{F}_{JI}^a \sim \mathbf{F}_{JI}^T - \mathbf{F}_{JI}$ of $\mathbf{F}_{JI}$ admits of an important geometrical interpretation. Since it is a singular, skew-symmetric 3×3 matrix we can express it with the aid of its null vector $\mathbf{f}_{JI}$ as $\mathbf{F}_{JI}^a \sim [\mathbf{f}_{JI}]_\times$. The point $\mathbf{f}_{JI}$ will then be the intersection of the two tangential lines at the epipoles (cf. Fig. 1).

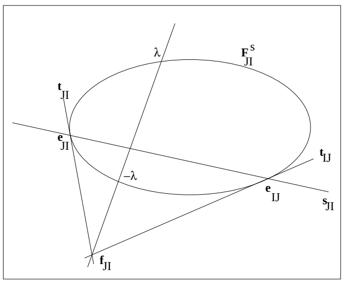

Fig. 1. The image of the horopter is in the general case a conic that is expressed by the symmetric part $\mathbf{F}_{JI}^s = \mathbf{F}_{JI} + \mathbf{F}_{JI}^T$ of the fundamental matrix and contains the epipoles. The antisymmetric part $\mathbf{F}_{JI}^a = \mathbf{F}_{JI}^T - \mathbf{F}_{JI} = [\mathbf{f}_{JI}]_\times$ of the fundamental matrix is expressed by the point $\mathbf{f}_{JI}$. Lines through the point $\mathbf{f}_{JI}$ intersect the image of the horopter in points parameterised by λ and $-\lambda$ if the canonical parametric equation introduced below is used.

4.1 Canonical parametric equation

To develop the theory further we introduce a suitably normalised canonical parametric representation of the image of the horopter through the following steps:

- Define the vector $\mathbf{f}_{JI}$ (with the resulting scale) through $[\mathbf{f}_{JI}]_\times := \mathbf{F}_{JI}^T - \mathbf{F}_{JI}$.
- Compute $\mathbf{f}_{JI}^T \mathbf{F}_{JI} \mathbf{f}_{JI} =: a$ and normalise $\mathbf{F}_{JI}$ and $\mathbf{f}_{JI}$ by the real third root of $-a$, i.e. set $\mathbf{F}_{JI} \rightarrow \mathbf{F}_{JI}/(-a)^{1/3}$ and $\mathbf{f}_{JI} \rightarrow \mathbf{f}_{JI}/(-a)^{1/3}$. We will then have $\mathbf{f}_{JI}^T \mathbf{F}_{JI} \mathbf{f}_{JI} = -1$ and $[\mathbf{f}_{JI}]_\times = \mathbf{F}_{JI}^T - \mathbf{F}_{JI}$ as before.
- Define $\mathbf{F}_{JI}^s$ through $\mathbf{F}_{JI}^s := \mathbf{F}_{JI} + \mathbf{F}_{JI}^T$.

- Normalise $||\mathbf{e}_{JI}||$ to unity and compute the scale of $\mathbf{e}_{IJ}$ such that the equation $\mathbf{e}_{IJ}^T \mathbf{F}_{JI} \mathbf{e}_{JI} = 1$ is valid.
- Define the matrix $\mathbf{H}_{JI}$ through $\mathbf{H}_{JI} := (\mathbf{e}_{JI}; \mathbf{f}_{JI}; \mathbf{e}_{IJ})$.

It is then easy to show that the following relations will be valid:

$$\mathbf{H}_{JI}^T \mathbf{F}_{JI}^s \mathbf{H}_{JI} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix} =: \mathbf{G}, \quad \mathbf{F}_{JI} \mathbf{e}_{JI} = \mathbf{e}_{JI} \times \mathbf{f}_{JI}, \quad \mathbf{F}_{JI} \mathbf{f}_{JI} = \mathbf{e}_{JI} \times \mathbf{e}_{IJ}.$$

All these relations are easy to verify. The defined matrix $\mathbf{H}_{JI}$ offers an easy to handle parametric representation of the image of the horopter as well as a transformation of the Kruppa equations to a very simple new normal form. To see the first property consider with any real number λ the vector $\boldsymbol{\lambda} := (\lambda^2, \lambda, 1)^T$. From the equation $\boldsymbol{\lambda}^T \mathbf{G} \boldsymbol{\lambda} = 0 \; \forall \lambda$ we deduce the relation $\boldsymbol{\lambda}^T \mathbf{H}_{JI}^T \mathbf{F}_{JI}^s \mathbf{H}_{JI} \boldsymbol{\lambda} = 0$ which expresses the fact that points on the image of the horopter are given by $\mathbf{H}_{JI} \boldsymbol{\lambda}$. In other words, the number λ is a projective parameter for the points on the image of the horopter. The epipole $\mathbf{e}_{IJ}$ is parameterised by $\lambda = 0$ and the epipole $\mathbf{e}_{JI}$ by $\lambda \to \infty$. Furthermore, since the vectors $(\lambda^2, \lambda, 1)$, $(\lambda^2, -\lambda, 1)$ and $(0, 1, 0)$ are linearly dependent, the points parameterised by λ and $-\lambda$ must be collinear with the point $\mathbf{f}_{JI}$ (see Fig. 1).

5 Simplifying the Kruppa equations

Now we come to the most important property of the matrix $\mathbf{H}_{JI}$ in simplifying the Kruppa equations. We reconsider these equations in the form $\mathbf{F}_{JI} \mathbf{X} \mathbf{F}_{JI}^T \sim [\mathbf{e}_{JI}]_\times \mathbf{X} [\mathbf{e}_{JI}]_\times$ and introduce the following transformation of the unknown matrix $\mathbf{X}$ to another symmetric matrix $\mathbf{Y}$: $\mathbf{X} = \mathbf{H}_{JI} \mathbf{Y} \mathbf{H}_{JI}^T$. Since $\mathbf{H}_{JI}$ is in the general case regular this transformation is one to one. From the normalisations defined above we will have: $\mathbf{F}_{JI} \mathbf{H}_{JI} = (\mathbf{e}_{JI} \times \mathbf{f}_{JI}; \mathbf{e}_{JI} \times \mathbf{e}_{IJ}; \mathbf{0}) =: (\mathbf{t}_{JI}; \mathbf{s}_{JI}; \mathbf{0})$ and $[\mathbf{e}_{JI}]_\times \mathbf{H}_{JI} = (\mathbf{0}; \mathbf{e}_{JI} \times \mathbf{f}_{JI}; \mathbf{e}_{JI} \times \mathbf{e}_{IJ}) = (\mathbf{0}; \mathbf{t}_{JI}; \mathbf{s}_{JI})$ with $\mathbf{t}_{JI}$ denoting the tangent at the epipole $\mathbf{e}_{JI}$ and $\mathbf{s}_{JI}$ the line connecting the two epipoles. We observe that the products $\mathbf{F}_{JI} \mathbf{H}_{JI}$ and $[\mathbf{e}_{JI}]_\times \mathbf{H}_{JI}$ differ only in a cyclical shift of the columns. $\mathbf{Y}$ must now solve the transformed Kruppa equations $(\mathbf{t}_{JI}; \mathbf{s}_{JI}; \mathbf{0}) \mathbf{Y} (\mathbf{t}_{JI}; \mathbf{s}_{JI}; \mathbf{0})^T \sim (\mathbf{0}; \mathbf{t}_{JI}; \mathbf{s}_{JI}) \mathbf{Y} (\mathbf{0}; \mathbf{t}_{JI}; \mathbf{s}_{JI})^T$. With $\mathbf{Y} \sim \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$ symmetric, the equations above translate to $a \mathbf{t}_{JI} \mathbf{t}_{JI}^T + b(\mathbf{t}_{JI} \mathbf{s}_{JI}^T + \mathbf{s}_{JI} \mathbf{t}_{JI}^T) + d \mathbf{s}_{JI} \mathbf{s}_{JI}^T = \lambda (d \mathbf{t}_{JI} \mathbf{t}_{JI}^T + e(\mathbf{t}_{JI} \mathbf{s}_{JI}^T + \mathbf{s}_{JI} \mathbf{t}_{JI}^T) + f \mathbf{s}_{JI} \mathbf{s}_{JI}^T)$ whence since the three matrices $\mathbf{t}_{JI} \mathbf{t}_{JI}^T$, $(\mathbf{t}_{JI} \mathbf{s}_{JI}^T + \mathbf{s}_{JI} \mathbf{t}_{JI}^T)$ and $\mathbf{s}_{JI} \mathbf{s}_{JI}^T$ are linearly independent we obtain: $a = \lambda d = \lambda^2 f$, $b = \lambda e$ and $d = \lambda f$. Renaming the parameters above we deduce that the matrix $\mathbf{Y}$ must be of the form $\mathbf{Y} \sim \begin{pmatrix} a\lambda^2 & b\lambda & c \\ b\lambda & a\lambda & b \\ c & b & a \end{pmatrix}$.

5.1 Incorporating the positivity constraint

An important property that must have a solution $\mathbf{X}$ of the Kruppa equations we are interested in is that of being positive definite. Otherwise, the Cholesky factorisation will fail to yield a real upper triangular calibration matrix $\mathbf{K}$ with $\mathbf{KK}^T = \mathbf{X}$. This property must be shared by the transformed solution $\mathbf{Y}$ as well and is usually very difficult to incorporate into numerical schemes that are searching in a multidimensional space for solutions of the Kruppa equations. Positive definitness means that all eigenvalues of $\mathbf{Y}$ must be positive. An easy analysis of these eigenvalues (for example using Hurwitz criteria) first reveals that necessary conditions for all eigenvalues of $\mathbf{Y}$ to be positive are: $\lambda > 0$ and $a > 0$. Using this and again renaming parameters, we arrive at $\mathbf{Y} \sim diag(\lambda, \sqrt{\lambda}, 1) \underbrace{\begin{pmatrix} 1 & a & b \\ a & 1 & a \\ b & a & 1 \end{pmatrix}}_{=:\mathbf{Z}} diag(\lambda, \sqrt{\lambda}, 1)$. From this we deduce that any solution $\mathbf{X}$ of the Kruppa equations that could be factored like $\mathbf{KK}^T$ should read as follows:

$$\mathbf{X} \sim \mathbf{KK}^T \sim \mathbf{H}_{JI} diag(\lambda, \sqrt{\lambda}, 1) \mathbf{Z} diag(\lambda, \sqrt{\lambda}, 1) \mathbf{H}_{JI}^T \ . \tag{3}$$

Now positivity of $\mathbf{X}$ is tantamount to positivity of the matrix $\mathbf{Z}$ with the eigenvalues $x_1 = 1 - b$, $x_{2/3} = (2 + b \mp \sqrt{8a^2 + b^2})/2$. By demanding $x_1 > 0$ and $x_2 > 0$ we thus arrive at the following remarkably simple result:

Proposition 1 (Region of positivity) *The matrix* $\mathbf{Z}$ *and consequently also a solution* $\mathbf{X} \sim \mathbf{H}_{JI} diag(\lambda, \sqrt{\lambda}, 1) \mathbf{Z} diag(\lambda, \sqrt{\lambda}, 1) \mathbf{H}_{JI}^T$ *of the Kruppa equations will be positive definite if and only if* $2a^2 - 1 < b < 1$.

Eq. (3) describes the Kruppa matrix $\mathbf{KK}^T$ with only three parameters, namely λ, a and b. That means, two parameters out of the five have been eliminated by using the two independent Kruppa equations. Moreover, besides $\lambda > 0$ the parameters a and b have to lie in a closed convex region of the $a-b$ space limited from above by the line $b = 1$ and from below by the parabola $b = 2a^2 - 1$.

5.2 The 3-d solution subspaces

Multiplying out the matrix $\mathbf{Y}$ and again absorbing the factor $\sqrt{\lambda}$ in a and the factor λ in b we also have the expression

$$\mathbf{KK}^T \sim \mathbf{H}_{JI} \left[\begin{pmatrix} \lambda^2 & & \\ & \lambda & \\ & & 1 \end{pmatrix} + a \begin{pmatrix} & \lambda & \\ \lambda & & 1 \\ & 1 & \end{pmatrix} + b \begin{pmatrix} & & 1 \\ & & \\ 1 & & \end{pmatrix} \right] \mathbf{H}_{JI}^T \ . \tag{4}$$

This expression clearly indicates that the Kruppa matrix $\mathbf{KK}^T$ is contained in a three-dimensional linear subspace. The parameter λ specifies the linear subspace and the vector $(1, a, b)^T$ specifies projective position within the subspace. As already mentioned, the parameter λ can also be given a geometric interpretation in terms of the image of the horopter. Indeed, it is not hard to see that the 3-d subspace above is also spanned by the matrices $\mathbf{p}_+(\lambda)\mathbf{p}_+(\lambda)^T$, $\mathbf{p}_-(\lambda)\mathbf{p}_-(\lambda)^T$ and $\mathbf{X}_0$ where $\mathbf{p}_\pm(\lambda) = \mathbf{H}_{JI}(\lambda, \pm\sqrt{\lambda}, 1)^T$ is a generating point pair on the image of the horopter parameterised by λ.

In the next section we introduce a second and a third image pair and outline how the new four Kruppa equations can be used in order to solve for the three parameters λ, a and b.

6 Solving the Kruppa equations from three image pairs

We now consider the Kruppa equations (2) arising from the second image pair IK using at the same time the SVD of the corresponding fundamental matrix $\mathbf{F}_{KI}$: $\mathbf{F}_{KI} \sim \mathbf{U}_{KI} diag(s_{IK}, 1, 0)\mathbf{U}_{IK}^T$ where $\mathbf{U}_{KI}$ and $\mathbf{U}_{IK}$ are orthogonal matrices, s_{IK} is the ratio of the two nonzero singular values of $\mathbf{F}_{KI}$ and the epipole $\mathbf{e}_{KI}$ is given by the third column $\mathbf{u}_{KI}^3$ of $\mathbf{U}_{KI}$ (cf. [4], [8]). It is then easy to show that the Kruppa equations take the form $(\mathbf{u}_{IK}^1 s_{IK}; \mathbf{u}_{IK}^2)^T \mathbf{K}\mathbf{K}^T (\mathbf{u}_{IK}^1 s_{IK}; \mathbf{u}_{IK}^2) \sim (\mathbf{u}_{KI}^2; -\mathbf{u}_{KI}^1)^T \mathbf{K}\mathbf{K}^T (\mathbf{u}_{KI}^2; -\mathbf{u}_{KI}^1)$. This is a homogeneous, two by two symmetrical matrix equation that is obviously equivalent to two scalar equations. Now upon substituting for $\mathbf{K}\mathbf{K}^T$ the reduced Kruppa matrix derived in (4) and elaborating, we arrive at an eigenvector equation of the following form: $\mathbf{P}(\lambda)\mathbf{a} \sim \mathbf{a}$. Here denotes $\mathbf{a}$ the vector $(1, a, b)^T$ and $\mathbf{P}(\lambda)$ is a 3×3 matrix whose entries are polynomials in λ with known coefficients. The degrees of the polynomials in the first row are 3, 2 and 1, in the second row 4, 3 and 2 and in the third row 5, 4 and 3. Similarly, the third image pair JK yields in the same way an eigenvector equation $\mathbf{Q}(\lambda)\mathbf{a} \sim \mathbf{a}$ and the problem of determining the parameter λ boils down to the condition for the two matrices $\mathbf{P}(\lambda)$ and $\mathbf{Q}(\lambda)$ to possess a common eigenvector. A necessary condition for that is the vanishing of the determinant of the commutator between $\mathbf{P}$ and $\mathbf{Q}$: $det(\mathbf{PQ} - \mathbf{QP}) = 0$. For three by three matrices the determinant of the commutator may be given by the expression $vec(\mathbf{Q})^T(\mathbf{P}^T \otimes \mathbf{P}^* - \mathbf{P}^{*T} \otimes \mathbf{P})vec(\mathbf{Q}^{*T})$ where vec denotes the row-wise conversion of a matrix to a column vector, $\otimes$ is the Kronecker product and $\mathbf{P}^*$ and $\mathbf{Q}^*$ are the adjoints of $\mathbf{P}$ and $\mathbf{Q}$. Fortunately, we could show that in each polynomial entry of the adjoints, as computed from the entries of $\mathbf{P}(\lambda)$ and $\mathbf{Q}(\lambda)$, the first and the last coefficient vanishes. That gives the following degrees of the entries of the adjoints: First row: 4, 3, 2; second row: 5, 4, 3; and third row: 6, 5, 4. Therefore, the expression for the determinant of the commutator given above results in a polynomial of 14-th degree in λ. Numerical experiments indicate that generally there will be at most only a few (four or six) real, positive roots of this polynomial. For each one of them we have to check whether $\mathbf{P}(\lambda)$ and $\mathbf{Q}(\lambda)$ possess common eigenvectors. Generally, this will be the case for only one root. The common eigenvector $\mathbf{a}$ will then contain the parameters a and b and eq. (4) will give the Kruppa matrix.

7 Conclusion and Outlook

In this paper we have studied in some detail the space of solutions of the Kruppa equations arising from a single image pair. We have revealed a very interesting connection with the image of the horopter and used them to considerably simplify the equations resulting in a closed formula for the Kruppa matrix depending on only three parameters. As a potential application of this simplification, we have reduced the six nonlinear Kruppa equations given three fundamental matrices to a polynomial of 14-th degree in one variable. Generally, all multiple solutions up to one are discarded from consistency conditions. It is of course needless to say that all previously reported ambiguous or degenerate situations (cf. [14],

[10]) remain valid. Such situations will manifest themselves in the framework of the present paper for example in giving a degenerate image of the horopter ($\mathbf{H}_{JI}$ singular). Currently we are investigating the possibility of factorising analytically the obtained polynomial of 14-th degree. That would reduce its degree even more and increase the stability of the resulting solutions.

References

1. O.D. Faugeras and Q-T. Luong. *The Geometry of Multiple Images.* MIT Press, 2001.
2. O.D. Faugeras, Q.-T. Luong, and S.J. Maybank. Camera Self-Calibration: Theory and Experiments. In G. Sandini, editor, *Proceedings of the 2nd European Conference on Computer Vision, Santa Margherita Ligure, Italy, volume 588 of Lecture Notes in Computer Science*, pages 321-334. Springer-Verlag, 1992.
3. R.I. Hartley. Self-Calibration from Multiple Views with a Rotating Camera. In J.-O. Eklundh, editor, *Proceedings of the 3rd European Conference on Computer Vision, Stockholm, Sweden, volume 800 of Lecture Notes in Computer Science*, pages 471-478. Springer-Verlag, 1994.
4. R.I. Hartley. Kruppa's Equations Derived from the Fundamental Matrix. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 19(2):133-135, 1997.
5. R.I. Hartley and A. Zisserman. *Multiple View Geometry in Computer Vision.* Cambridge University Press, 2000.
6. E. Kruppa. Zur Ermittlung eines Objektes aus zwei Perspektiven mit innerer Orientierung. *Sitz.-Ber. Akad. Wiss., Wien, math. naturw. Abt. IIa., vol. 122*, pages 1939-1948, 1913.
7. D. Liebowitz and A. Zisserman. Combining Scene and Auto-calibration Constraints. In *Proceedings of the 7th International Conference on Computer Vision, Kerkyra, Greece*, pages 293-300. IEEE Computer Society Press, 1999.
8. M.I. Lourakis and R. Deriche. Camera Self-Calibration Using the Singular Value Decomposition of the Fundamental Matrix. In Proceedings of the 4th Asian Conference on Computer Vision, volume I, pages 403-408, 2000.
9. Q.-T. Luong and O.D. Faugeras. Self-Calibration of a Moving Camera from point Correspondences and Fundamental Matrices. *International Journal of Computer Vision*, 22(3):261-289, 1997.
10. Y. Ma, R. Vidal, J. Kosecka, and S. Sastry. Kruppa Equation Revisited: Its Renormalization and Degeneracy. In D. Vernon, editor, *Proceedings of the 6th European Conference on Computer Vision, Dublin, Ireland, volume 1843 of Lecture Notes in Computer Science*, pages II.562-II.577. Springer-Verlag, 2000.
11. S.J. Maybank and O.D. Faugeras. A Theory of Self Calibration of a Moving Camera. *International Journal of Computer Vision*, 8(2):123-151, 1992.
12. M. Pollefeys and L. Van Gool. Stratified Self-Calibration with the Modulus Constraint. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 21(8):707-724, 1999.
13. J.G. Semple and G.T. Kneebone. *Algebraic Projective Geometry.* Oxford University Press, 1952.
14. P. Sturm. A Case Against Kruppa's Equations for Camera Self-Calibration. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 22(10):1199-1204, 2000.
15. B. Triggs *The Absolute Quadric* In *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition*, pages 609-614, 1997.
16. C. Zeller and O. D. Faugeras. *Camera Self-Calibration from Video Sequences: The Kruppa Equations Revisited* Research Report 2793, INRIA, 1996.