Complete Moment Invariants and Pose Determination for Orthogonal Transformations of 3D Objects

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Abstract. It is well known that for the simpler problem of constructing translation invariants of grey scale images (1D, 2D or 3D) central moments can be used. There are plain closed formulae expressing them in terms of the ordinary geometrical moments. Moreover, central moments *are* ordinary moments of the properly normalized image.

In this paper we present moment invariants for the more involved problem of rotations and reflections of 3D density objects, having exactly the same qualities as those mentioned above of central moments.

The mathematical analysis of this problem is complicated mainly due to noncommutativity of the group of 3D rotations $SO(3)$. However, by constructing basis functions using harmonic polynomials, rather than monomials, we achieve a decomposition of the action of $SO(3)$ in irreducible representations acting on invariant subspaces, thus simplifying the problem.

Using a suitable generating function for harmonic polynomials we work out a novel and very compact description of these subspaces. In addition, we introduce the notion of "spherical moments" denoting inner products of the basis functions with an object, and we encode them using the same generating function.

In conjunction with the Cayley-Klein parameterization of $SO(3)$ we obtain a simple relationship between the encoded spherical moments of two rotated/reflected versions of a 3D object. This relationship enables us to express the spherical moments of a uniquely normalized object in terms of the spherical moments of the not normalized (actual) object, just as we can express central moments in terms of ordinary moments. In doing so we don't lose any information and since moments uniquely characterize an object with compact support we see that we have constructed complete moment invariants.

The normalization process itself is carried out using exclusively moments of third order and yields at the same time unique pose determination.

Keywords: 3D moment invariants, completeness, irreducible representations, harmonic polynomials, spherical moments, 3D pose determination.

1 Introduction

Modern imaging instrumentation is capable of providing images with complete interior 3D detail. Examples are medical diagnosis systems based on computerized tomography (CT), magnetic resonance imaging (MRI), positron emitting tomography (PET) as well as active range finders, stereoscopic backprojection etc. In all these instances automatic registration and recognition demands the extraction of features which should be invariant to an arbitrary 3D motion of the object.

The focus of this paper is on deriving for that problem closed analytical formulae for a complete set of independent invariant features based on image moments as well as efficiently solving the positioning problem. For that purpose we develop a normalization scheme which uses only moments of third order.

It turns out that publications on this topic using moment techniques have been rather sparse in the last two decades. This is in contrast to the corresponding 2D problems where publications abound.

Dirilten and Newman [4] have utilized the method of contracting indices of moment tensors. The resulting invariants are not complete and may even be not independent. Similar remarks apply also for the tensor and matrix techniques used in [3] and [12].

Sadjadi and Hall [11] have attempted to generalize results of the theory of 2D moment invariants. However, only second order moment invariants have been explicitly derived.

Using tensor algebra Faber and Stokely [6] could estimate an affine transformation lying between two medical objects known to be similar. Even so, they need fifth order moments which may be more vulnerable to noise contamination than lower order moments. For a comprehensive study of the behaviour of various 2D moments in the presence of noise cf. [13].

Finally, Lo and Don [10] have recognized the need for invoking the representations of the group $SO(3)$. Again, only low order invariants are derived (≤ 3) without pursuing completeness.

We present here a self contained closed mathematical solution for the problem of unique pose determination and computation of complete moment invariants for general orthogonal transformations of 3D objects. We assume that the object is lying entirely within a compact region of the 3D Euclidean space and that it is described through its voxel intensity representation well separated from the background. However, point sets or sets of line segments can be considered too within this framework by modelling them as sums of Dirac distributions. The pose determination is carried out through a unique normalization procedure. Therefore, no known point correspondences are required.

The paper is organized as follows: In Sect. 2 we summarize the needed mathematical background concerning harmonic polynomials, spherical moments and representations of the group $SO(3)$. In Sect. 3 we develop the concept of ζ-coding which simplifies the analysis considerably. Section 4 deals with the normalization procedure and the unique pose determination and in Sect. 5 we formulate explicitly the complete invariants. Section 6 contains our concluding remarks.

2 Preliminaries

Basic facts contained in this section can be found for example in [8] or [5].

2.1 Harmonic polynomials and spherical moments

The subspace

$$Q_n := < x^n, x^{n-1}y, \ldots, x^p y^q z^r, \ldots, z^n > \quad ; \quad p+q+r=n \ ,$$

spanned by all n-th order monomials in three cartesian variables x, y and z is obviously invariant with respect to every linear transformation $\boldsymbol{R}$; i.e. if $q_n(\boldsymbol{x}) \in Q_n$ then $q_n(\boldsymbol{Rx}) \in Q_n$. The dimension of Q_n is $\frac{(n+1)(n+2)}{2}$. Now, it is well known that if we restrict $\boldsymbol{R}$ to be orthogonal, $\boldsymbol{R} \in O(3)$, then we can find within Q_n further invariant subspaces which are irreducible. We denote these subspaces by M_{nl} and obtain a decomposition of Q_n in a direct sum

$$Q_n = M_{nn} \oplus M_{n,n-2} \oplus M_{n,n-4} \oplus \cdots \oplus \begin{cases} M_{n0} \text{ if } n \text{ even} \\ M_{n1} \text{ if } n \text{ odd} \end{cases} .$$

The basis elements of M_{nl} are of the form

$$|\boldsymbol{x}|^n Y_l^m(\vartheta,\phi) = |\boldsymbol{x}|^{l+2d} Y_l^m(\vartheta,\phi) = |\boldsymbol{x}|^{2d} e_l^m(\boldsymbol{x}) = |\boldsymbol{x}|^{n-l} e_l^m(\boldsymbol{x}) \ \text{, where}$$

$$\boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = |\boldsymbol{x}| \cdot \begin{pmatrix} \sin\vartheta\cos\phi \\ \sin\vartheta\sin\phi \\ \cos\vartheta \end{pmatrix} \ , \tag{1}$$

$Y_l^m(\vartheta,\phi)$ are spherical harmonics of degree l,

$$e_l^m(\boldsymbol{x}) = |\boldsymbol{x}|^l Y_l^m(\vartheta,\phi) \tag{2}$$

are l-th order homogeneous harmonic polynomials and $n-l=2d$ are nonnegative even integers. The dimension of M_{nl} is $2l+1$ and we can verify

$$\begin{aligned} &\dim(M_{nn}) + \dim(M_{n,n-2}) + \cdots + \dim(M_{n0} \text{ or } M_{n1}) = \\ &= (2n+1) + (2n-3) + (2n-7) + \cdots + (1 \text{ or } 3) = \\ &= \frac{(n+1)(n+2)}{2} \ (\text{ in both cases }) \ = \dim(Q_n) \ . \end{aligned}$$

We use indices m running from $+l$ to $-l$ and ask for an explicit expression for the spherical harmonics $Y_l^m(\vartheta,\phi)$. The result most frequently reported in the vision literature ([8], [9]) is based on the Rodrigues formula for the associated Legendre functions $P_l^m(\vartheta)$

$$Y_l^m(\vartheta,\phi) = P_l^m(\vartheta) \cdot e^{jm\phi} \quad \text{and} \tag{3}$$

$$P_l^m(\vartheta) \sim (j\sin\vartheta)^m \cdot \frac{d^{l+m}}{du^{l+m}}[(u^2-1)^l]\Big|_{u=\cos\vartheta} \ ,$$

where $j^2 = -1$. We shall make use of the sign "$\sim$" instead of "$=$" every time we are neglecting constant factors. Now, the point is that this expression for spherical harmonics is of little value if we want to study how are harmonic polynomials changing under the action of a rotation. Of course, there is the representation law stating that if we collect all l-th order harmonic polynomials in a $(2l+1)$-dimensional vector $\boldsymbol{e}_l(\boldsymbol{x}) := \left(e_l^l(\boldsymbol{x}), e_l^{l-1}(\boldsymbol{x}), \cdots, e_l^{-l}(\boldsymbol{x})\right)^T$ then we have [5]

$$\boldsymbol{e}_l(\boldsymbol{P}\boldsymbol{x}) = \boldsymbol{o}_l(\boldsymbol{P})\boldsymbol{e}_l(\boldsymbol{x}) \quad ; \quad \boldsymbol{P} \in SO(3) \tag{4}$$

with $\boldsymbol{o}_l(\boldsymbol{P})$ being a $(2l+1)\times(2l+1)$ unitary matrix if the $e_l^m(\boldsymbol{x})$'s are normalized to be orthonormal on the unit sphere. But now we need the matrix elements of $\boldsymbol{o}_l(\boldsymbol{P})$ in dependence of the rotation parameters and we have not been able to find in the literature a really succinct description for that.
Not to be discouraged we can help things if we start from an integral formula for the associated Legendre functions $P_l^m(\vartheta)$ found in [5] or [14]:

$$P_l^m(\vartheta) \sim \frac{1}{2\pi}\int_0^{2\pi} (\cos\vartheta + j\sin\vartheta\cos\gamma)^l e^{-jm\gamma}\, d\gamma \ .$$

Expanding the binomial expression and integrating the appearing trigonometric functions using Euler's formula yields in conjunction with (1), (2) and (3)

$$e_l^m(\boldsymbol{x}) \sim \left(\frac{x+jy}{2}\right)^m (-jz)^{l-m} \sum_\mu \binom{l}{\mu}\binom{l-\mu}{m+\mu}\left(-\frac{x^2+y^2}{4z^2}\right)^\mu =: \hat{e}_l^m(\boldsymbol{x}) \ . \tag{5}$$

The sums are extended over all indices for which all appearing combinatorial symbols $\binom{n}{\nu}$ are defined, i.e. $0 \le \nu \le n$. In (5) we have defined on the right side modified harmonic polynomials $\hat{e}_l^m(\boldsymbol{x})$ that are orthogonal but not orthonormal on the unit sphere. It can be shown that the connection to the orthonormal harmonic polynomials $e_l^m(\boldsymbol{x})$ is given by $e_l^m(\boldsymbol{x}) = c_l^m \cdot \hat{e}_l^m(\boldsymbol{x})$ where

$$c_l^m := \frac{\sqrt{(2l+1)(l+m)!(l-m)!}}{l!} = c_l^{-m} \ . \tag{6}$$

We note here the symmetry relation of the so represented harmonic polynomials:

$$e_l^{-m}(\boldsymbol{x}) = (-1)^{l-m}\left[e_l^m(\boldsymbol{x})\right]^* \ . \tag{7}$$

Inserting $m = 0$ we obtain from above $e_l^0(\boldsymbol{x})^* = (-1)^l e_l^0(\boldsymbol{x})$ i.e. if l is even $e_l^0(\boldsymbol{x})$ is real and if l is odd $e_l^0(\boldsymbol{x})$ is imaginary. More details about these facts can be found in [1].
We now define "spherical moments" F_{nl}^m being inner products of a function $f(\boldsymbol{x}) \in (\mathbb{R}^+)^{\mathbb{R}^3}$ with the basis functions of M_{nl}

$$F_{nl}^m := \int_{\mathbb{R}^3} f(\boldsymbol{x})|\boldsymbol{x}|^{n-l} e_l^m(\boldsymbol{x})^*\, d\boldsymbol{x} \tag{8}$$

and collect all spherical moments of order n and degree $l = n - 2d$ in a $(2l+1)$-dimensional vector

$$\boldsymbol{F}_{nl} := \left(F_{nl}^{l}, F_{nl}^{l-1}, \cdots, F_{nl}^{-l}\right)^T = \int_{\mathbb{R}^3} f(\boldsymbol{x}) |\boldsymbol{x}|^{n-l} \boldsymbol{e}_l(\boldsymbol{x})^* \, d\boldsymbol{x} \ . \tag{9}$$

Due to the representation law (4) we can compute the effect of a rotation of $f(\boldsymbol{x})$ on $\boldsymbol{F}_{nl}$: $(\boldsymbol{F}_{\boldsymbol{P}})_{nl} := \int_{\mathbb{R}^3} f(\boldsymbol{P}^{-1}\boldsymbol{x}) |\boldsymbol{x}|^{n-l} \boldsymbol{e}_l(\boldsymbol{x})^* \ d\boldsymbol{x}, \quad \boldsymbol{P} \in SO(3)$. Since $|\boldsymbol{P}\boldsymbol{x}| = |\boldsymbol{x}|$ we obtain from (4)

$$(\boldsymbol{F}_{\boldsymbol{P}})_{nl} = \boldsymbol{o}_l(\boldsymbol{P})^* \boldsymbol{F}_{nl} \ . \tag{10}$$

This is already a great simplification of our problem since we have found the smallest possible multiplets consisting of spherical moments that are transformed exclusively among themselves under a rotation of the object. Very useful in this context appears to be (5), which can be used to express spherical moments as linear combinations of geometrical moments of the same order:

$$\begin{aligned} \hat{F}_{nl}^{m} = \frac{j^{l-m}}{2^m} \sum_{\mu} \left(-\frac{1}{4}\right)^{\mu} \binom{l}{\mu} \binom{l-\mu}{m+\mu} \cdot \\ \cdot \int_{\mathbb{R}^3} f(\boldsymbol{x}) (x^2+y^2+z^2)^{\frac{n-l}{2}} (x^2+y^2)^{\mu} (x-jy)^m z^{l-m-2\mu} \, d\boldsymbol{x}. \end{aligned} \tag{11}$$

Note that the above integral and hence the whole expression (11) too is a linear combination of n-th order geometrical moments. Thus, we can compute from the geometrical moments spherical moments of any order and degree (and vice versa). If we recall that the representation matrices $\boldsymbol{o}_l(\boldsymbol{P})$ in (4) are unitary we see from (10) that the norm of $(\boldsymbol{F}_{\boldsymbol{P}})_{nl}$ is invariant w.r.t. rotations:

$$|(\boldsymbol{F}_{\boldsymbol{P}})_{nl}| = |\boldsymbol{F}_{nl}| \ . \tag{12}$$

However, this classical system of invariants is not complete since we get for every $(2l+1)$-dimensional subspace M_{nl} only one single invariant. In the next section we will look more deeply into the structure of the system of harmonic polynomials.

3 The ζ-coding

3.1 Definition

For every nonnegative l we define the $(2l+1)$-dimensional vector

$$\boldsymbol{p}_l(\zeta) := \left(\zeta^l, \zeta^{l-1}, \cdots, \zeta^{-l}\right)^T \quad , \quad \zeta \in \mathbb{C} \ , \tag{13}$$

and form the product

$$\hat{e}_l(\boldsymbol{x}; \zeta) := \boldsymbol{p}_l(\zeta)^T \hat{\boldsymbol{e}}_l(\boldsymbol{x}) = \sum_{m=-l}^{l} \hat{e}_l^m(\boldsymbol{x}) \zeta^m \ . \tag{14}$$

Thus, we have encoded the *modified* harmonic polynomials of order l into the coefficients of a polynomial of order $2l$ in ζ divided by ζ^l. The result $\hat{e}_l(\boldsymbol{x};\zeta)$ which we shall call ζ-coding of harmonic polynomials is a generating function for the latter. Now it is straightforward to compute from (5) the above sum. After some algebra the lengthy expression collapses to give

$$\hat{e}_l(\boldsymbol{x};\zeta) = \left[\boldsymbol{p}_1(\zeta)^T \boldsymbol{S}\boldsymbol{x}\right]^l \tag{15}$$

with $\boldsymbol{S} := \frac{1}{2}\begin{pmatrix} 1 & j & 0 \\ 0 & 0 & -2j \\ 1 & -j & 0 \end{pmatrix}$ a constant matrix, and therefore

$$\hat{e}_l(\boldsymbol{x};\zeta) = \left[\hat{e}_1(\boldsymbol{x};\zeta)\right]^l \ . \tag{16}$$

The simplicity of this result is remarkable: The ζ-coding of the harmonic polynomials of order l equals the l-th power of the ζ- coding of the harmonic polynomials of first order. Stated differently, $\hat{e}_l^m(\boldsymbol{x})$ is the coefficient of ζ^m in the expansion of $\left[\hat{e}_1^1(\boldsymbol{x})\cdot\zeta + \hat{e}_1^0(\boldsymbol{x}) + \hat{e}_1^{-1}(\boldsymbol{x})\cdot\zeta^{-1}\right]^l$. The above relationships constitute a very compact description of the irreducible invariant subspaces of $SO(3)$. In the next subsection we will explore the effect of a rotation on the encoded harmonic polynomials.

3.2 ζ-coding and rotation matrices

The purpose of this subsection is to evaluate $\hat{e}_l(\boldsymbol{P}\boldsymbol{x};\zeta)\ ;\ \boldsymbol{P} \in SO(3)$. Due to (16) it suffices to consider $\hat{e}_1(\boldsymbol{P}\boldsymbol{x};\zeta)$. We first note that since we have from (15) $\hat{e}_1(\boldsymbol{x};\zeta) = \boldsymbol{p}_1(\zeta)^T\boldsymbol{S}\boldsymbol{x}$ and from the definition (14) $\hat{e}_1(\boldsymbol{x};\zeta) = \boldsymbol{p}_1(\zeta)^T\hat{\boldsymbol{e}}_1(\boldsymbol{x})$, it follows $\hat{\boldsymbol{e}}_1(\boldsymbol{x}) = \boldsymbol{S}\boldsymbol{x}$ and $\boldsymbol{x} = \boldsymbol{S}^{-1}\hat{\boldsymbol{e}}_1(\boldsymbol{x})$. Therefore

$$\hat{e}_1(\boldsymbol{P}\boldsymbol{x};\zeta) = \boldsymbol{p}_1(\zeta)^T\hat{\boldsymbol{e}}_1(\boldsymbol{P}\boldsymbol{x}) = \boldsymbol{p}_1(\zeta)^T\boldsymbol{S}\boldsymbol{P}\boldsymbol{x} = \boldsymbol{p}_1(\zeta)^T\boldsymbol{S}\boldsymbol{P}\boldsymbol{S}^{-1}\hat{\boldsymbol{e}}_1(\boldsymbol{x}) \ . \tag{17}$$

The question now arises about the most suitable parameterization for the rotation matrices $\boldsymbol{P}$. It turns out that very good services in this respect offers the Cayley-Klein parameterization. It may be obtained either by the stereographic projection or by the homomorphism of the group $SO(3)$ to the special unitary group $SU(2)$ [8]. Without further discussing here these concepts we confine ourselves in merely giving this parameterization in the following form:

$$\boldsymbol{P}(a,b) = \begin{pmatrix} \Re\{a^2+b^2\} & -\Im\{a^2-b^2\} & 2\Im\{ab\} \\ \Im\{a^2+b^2\} & \Re\{a^2-b^2\} & -2\Re\{ab\} \\ 2\Im\{ab^*\} & 2\Re\{ab^*\} & aa^*-bb^* \end{pmatrix} \ ;\ a,b \in \mathbb{C}\ ,\ aa^*+bb^* = 1 \ .$$

$\boldsymbol{P}$ represents a rotation with axis $(\Re\{b\},\Im\{b\},\Im\{a\})$ and angle $\arccos(2\Re\{a\}^2 - 1)$. It should be noted that the pairs (a,b) and $(-a,-b)$ yield the same rotation matrix. This ambiguity can easily be removed by considering only pairs (a,b)

with $\Re\{a\} \geq 0$. Now using this parameterization and the constant matrix $\boldsymbol{S}$ we evaluate $\boldsymbol{SPS}^{-1}$ and obtain

$$\boldsymbol{SPS}^{-1} = \begin{pmatrix} a^2 & ab & b^2 \\ -2ab^* & aa^* - bb^* & 2a^*b \\ (b^*)^2 & -a^*b^* & (a^*)^2 \end{pmatrix} \ .$$

Multiplicating this relationship on the left by $\boldsymbol{p}_1(\zeta)^T$ as required by (17) and using the definition (13) with $l = 1$ yields

$$\boldsymbol{p}_1(\zeta)^T \boldsymbol{SPS}^{-1} = \frac{(a\zeta - b^*)(b\zeta + a^*)}{\zeta} \cdot \boldsymbol{p}_1 \left(\frac{a\zeta - b^*}{b\zeta + a^*} \right)^T \ .$$

Therefore $\hat{e}_1(\boldsymbol{Px};\zeta) = \frac{(a\zeta-b^*)(b\zeta+a^*)}{\zeta} \cdot \boldsymbol{p}_1 \left(\frac{a\zeta-b^*}{b\zeta+a^*} \right)^T \hat{\boldsymbol{e}}_1(\boldsymbol{x})$ and with the definition (14) $\hat{e}_1(\boldsymbol{Px};\zeta) = \frac{(a\zeta-b^*)(b\zeta+a^*)}{\zeta} \cdot \hat{e}_1 \left(\boldsymbol{x}; \frac{a\zeta-b^*}{b\zeta+a^*} \right)$. Finally, (16) yields

$$\hat{e}_l(\boldsymbol{Px};\zeta) = \left[\frac{(a\zeta - b^*)(b\zeta + a^*)}{\zeta} \right]^l \cdot \hat{e}_l \left(\boldsymbol{x}; \frac{a\zeta - b^*}{b\zeta + a^*} \right) \ . \tag{18}$$

We generalize these results in such a way as to be able to cope with reflections too. Every 3×3 orthogonal matrix $\boldsymbol{R}$ can be written in the form

$$\boldsymbol{R} = \varepsilon \boldsymbol{P} \ \ ; \ \ \varepsilon \in \{+1, -1\} \ \ ; \ \ \boldsymbol{P} \in SO(3) \ , \ \boldsymbol{R} \in O(3) \ .$$

With the aid of (5) we can compute the result of reflecting harmonic polynomials on the origin:

$$\hat{e}_l^m(\varepsilon \boldsymbol{x}) = \varepsilon^l \hat{e}_l^m(\boldsymbol{x}) \ . \tag{19}$$

Together with (18) and the definition (14) this gives

$$\hat{e}_l(\boldsymbol{Rx};\zeta) = \left[\varepsilon \frac{(a\zeta - b^*)(b\zeta + a^*)}{\zeta} \right]^l \cdot \hat{e}_l \left(\boldsymbol{x}; \frac{a\zeta - b^*}{b\zeta + a^*} \right) \ . \tag{20}$$

Thus, the effect of a rotation/reflection upon the ζ-coded harmonic polynomials is essentially a linear fractional transformation of the variable ζ. This will enable us to determine position and to derive invariants by purely algebraic means.

3.3 ζ-coding and spherical moments

The concept of ζ-coding is readily transferred to spherical moments. We consider (9) with $\boldsymbol{e}_l(\boldsymbol{x})$ replaced by modified harmonic polynomials $\hat{\boldsymbol{e}}_l(\boldsymbol{x})$ and $\boldsymbol{F}_{nl}$ replaced by modified spherical moments $\hat{\boldsymbol{F}}_{nl}$:

$$\hat{\boldsymbol{F}}_{nl} := \left(\hat{F}_{nl}^l, \hat{F}_{nl}^{l-1}, \cdots, \hat{F}_{nl}^{-l} \right)^T = \int_{\mathbb{R}^3} f(\boldsymbol{x}) |\boldsymbol{x}|^{n-l} \hat{\boldsymbol{e}}_l(\boldsymbol{x})^* \, d\boldsymbol{x} \ . \tag{21}$$

Now we apply the definition (14) by analogy to the modified spherical moments

$$\hat{F}_{nl}(\zeta) := \boldsymbol{p}_l(\zeta)^T \hat{\boldsymbol{F}}_{nl} = \int_{\mathbb{R}^3} f(\boldsymbol{x})|\boldsymbol{x}|^{n-l}\hat{e}_l(\boldsymbol{x};\zeta^*)^*\, d\boldsymbol{x} \tag{22}$$

and examine the action of the orthogonal group $O(3)$ on the encoded modified spherical moments $\left(\hat{F}_{\boldsymbol{R}}\right)_{nl}(\zeta) := \int_{\mathbb{R}^3} f(\boldsymbol{R}^{-1}\boldsymbol{x})|\boldsymbol{x}|^{n-l}\hat{e}_l(\boldsymbol{x};\zeta^*)^*\, d\boldsymbol{x}$. Equation (20) gives

$$\begin{aligned}\left(\hat{F}_{\boldsymbol{R}}\right)_{nl}(\zeta) &= \int_{\mathbb{R}^3} f(\boldsymbol{x})|\boldsymbol{R}\boldsymbol{x}|^{n-l}\hat{e}_l(\boldsymbol{R}\boldsymbol{x};\zeta^*)^*\, d\boldsymbol{x} = \\ &= \left[\varepsilon\frac{(a^*\zeta - b)(b^*\zeta + a)}{\zeta}\right]^l \int_{\mathbb{R}^3} f(\boldsymbol{x})|\boldsymbol{x}|^{n-l}\hat{e}_l\left(\boldsymbol{x};\frac{a\zeta^* - b^*}{b\zeta^* + a^*}\right)^* d\boldsymbol{x}\end{aligned}$$

and with (22) we obtain the fundamental result

$$\left(\hat{F}_R\right)_{nl}(\zeta) = \left[\varepsilon\frac{(a^*\zeta - b)(b^*\zeta + a)}{\zeta}\right]^l \cdot \hat{F}_{nl}\left(\frac{a^*\zeta - b}{b^*\zeta + a}\right) \quad . \tag{23}$$

This equation describes very succinctly the way modified spherical moments are changing under the action of a proper or improper orthogonal transformation $\boldsymbol{R}$ with parameters (a, b, ε).

Let us now assume that $\hat{F}_{nl}^m$ are the modified spherical moments of an object lying in some standard position. $\left(\hat{F}_R\right)_{nl}^m$ will be the measured moments of the same object lying in a position described by the matrix $\boldsymbol{R}$ w.r.t the standard position. If we know the parameters of $\boldsymbol{R}$ we may obtain the moments $\hat{F}_{nl}^m$ (invariants) from the measurements $\left(\hat{F}_R\right)_{nl}^m$ by inverting (23). That demonstrates at the same time the power of the concept of ζ-coding. Applying the substitution $\zeta \to (a\zeta + b)/(-b^*\zeta + a^*)$ we get

$$\hat{F}_{nl}(\zeta) = \left[\varepsilon\frac{(a\zeta + b)(-b^*\zeta + a^*)}{\zeta}\right]^l \cdot \left(\hat{F}_R\right)_{nl}\left(\frac{a\zeta + b}{-b^*\zeta + a^*}\right) \quad . \tag{24}$$

We may parallel this result with the familiar formula expressing central moments in dependence of the measured moments and of the object's center of gravity describing position. As it is well known the latter is uniquely obtained from the measured moments by normalizing central moments of first order to zero. This imposition happens to correspond in that case with physical considerations. But this is not necessary. It could have been derived as well from the mathematical requirement of uniqueness alone.

The normalization procedure relevant to the problem discussed in this paper will be the subject of the next chapter. No physical aspects will be considered. This is in contrast to the very frequently proposed normalization using principal axes of the symmetric matrix of second moments. Due to ambiguities principal axes don't give unique normalization, so to resolve the ambiguities one has to resort to higher order moments anyway. Instead, for the normalization to be described in the next chapter only third order moments will be used. Moments of order two should be reserved for affine normalization (shearing and nonisotropic scaling). In fact, we have shown in [2] how to uniquely accomplish a reduction of an affine deformation to an orthogonal transformation by normalizing all moments of first and second order to specific standard values.

4 Normalization

Our starting point now is (24). We will derive a unique rotation/reflection with parameters a, b and ε depending only on measurements $\left(\hat{F}_R\right)_{3l}^{m}$ such that certain spherical moments of order three $\hat{F}_{3l}^{m}$ are normalized to specific standard values. That will be achieved in two steps. We first look at $n = 3$ and $l = 1$, i.e. spherical moments of third order and first degree and use the following abbreviations

$$u := \left(\hat{F}_R\right)_{31}^{1} \qquad jv := \left(\hat{F}_R\right)_{31}^{0} \qquad C := \sqrt{4uu^* + v^2}\ . \tag{25}$$

Here is C essentially the (nontrivial) invariant (12) of the subspace M_{31}: $\sqrt{3}C = |(\boldsymbol{F_R})_{31}| = |\boldsymbol{F}_{31}|$. That can be easily shown using (6) and (25). Also note that $\left(\hat{F}_R\right)_{31}^{-1} = u^*$ and $\left(\hat{F}_R\right)_{31}^{0*} = -\left(\hat{F}_R\right)_{31}^{0}$ i.e. $v \in I\!R$. We assume $C > 0$ excluding all images with $C = 0$. Clearly, the latter is a set of measure zero although it might contain important images exhibiting certain symmetries which would require a separate investigation. We now ask which rotation/reflection (24), if any, achieves

$$\hat{F}_{31}^{1} = \hat{F}_{31}^{-1} = 0 \qquad \text{and} \qquad \hat{F}_{31}^{0} = jC\ ? \tag{26}$$

To answer this question we expand (24) with $n = 3$ and $l = 1$ and obtain the equations

$$a^2u - jab^*v + (b^*)^2 u^* = 0 \qquad \text{and}$$
$$\varepsilon[2abu + j(aa^* - bb^*)v - 2a^*b^*u^*] = jC\ .$$

The first of the equations above is essentially a quadratic one in a/b^* with the solutions $a = j\frac{v\pm C}{2u}b^* = \pm j\frac{C\pm v}{2u}b^* =: j\sigma\frac{C+\sigma v}{2u}b^*$ where $\sigma \in \{+1, -1\}$ denotes the sign. Inserting into the second equation we obtain $bb^* = \sigma\varepsilon\frac{2uu^*}{C(C+\sigma v)} = \sigma\varepsilon\frac{C-\sigma v}{2C}$ and hence $aa^* = \sigma\varepsilon\frac{C+\sigma v}{2C}$. Since $C > 0$ and $C \pm v \geq 0$ we must have $\sigma\varepsilon = 1$ i.e. $\sigma = \varepsilon$. Finally, after some algebraic manipulations we obtain the intermediate result

$$a = \sqrt{\frac{C + \sigma v}{2C}}e^{j\alpha} \qquad \text{and} \qquad b = j\sigma\frac{u^*}{Ca} = j\sigma\sqrt{\frac{C - \sigma v}{2C}}e^{-j\alpha}e^{-j\arg(u)}$$

with $\sigma \in \{+1, -1\}$ and $\alpha \in I\!R \pmod{2\pi}$ yet to be computed from normalizing constraints to be imposed on the subspace M_{33}. To this end we note that parameters (a, b, ε) with arbitrary $\varepsilon = \sigma$ and α as above will give a rotation/reflection which satisfies the constraints (26). We therefore use as a first intermediate step the particularly simple pure rotation with the known parameters $(a' := \sqrt{\frac{C+v}{2C}}, b' := j\frac{u^*}{Ca'} = j\sqrt{\frac{C-v}{2C}}e^{-j\arg(u)}, \varepsilon' := +1)$. Denoting with $(G_R)_{nl}^m$ the spherical moments obtained after applying the above rotation to the actual object we get

$$(G_R)_{nl}(\zeta) := \left[\frac{(a'\zeta + b')(-b'^*\zeta + a')}{\zeta}\right]^l \cdot \left(\hat{F}_R\right)_{nl}\left(\frac{a'\zeta + b'}{-b'^*\zeta + a'}\right) \qquad (27)$$

with $(G_R)_{31}(\zeta) \equiv jC$.
To derive the second step we look at $n = 3$ and $l = 3$ and use the abbreviations

$$w := (G_R)_{33}^1 \Big/ \left|(G_R)_{33}^1\right| , |w| = 1, \quad q := (G_R)_{33}^2 , \quad \epsilon := \operatorname{sgn}\left(\Re\{qw^{*2}\}\right) \ .$$

It should be clear that $(G_R)_{33}^1$ and $(G_R)_{33}^2$ are obtained via (27) as linear combinations of the $\left(\hat{F}_R\right)_{33}^m$ with coefficients depending on a' and b' and are therefore known numbers. Hence, so are w q and ϵ. Now it is easily seen that, in order for the second rotation/reflection leading to the spherical moments $\hat{F}_{nl}^m$ (invariants) to preserve the normalization achieved in the first step it must be of the form

$$\hat{F}_{nl}(\zeta) = (G_R)_{nl}(\gamma e^{j2\phi}\zeta^\gamma),\ \gamma \in \{+1, -1\} \text{ such that } \hat{F}_{31}(\zeta) = (G_R)_{31}(\gamma e^{j2\phi}\zeta^\gamma) \equiv jC \ . \qquad (28)$$

The above corresponds to a rotation/reflection with parameters $(e^{j\phi}, 0, 1)$ if $\gamma = 1$ or $(0, e^{j\phi}, -1)$ if $\gamma = -1$. We claim that by choosing $\gamma = \epsilon$ and $e^{j2\phi} = \epsilon w^*$, i.e.

$$\hat{F}_{nl}(\zeta) = (G_R)_{nl}(w^*\zeta^\epsilon)$$

we achieve the final normalizations

$$\hat{F}_{33}^1 \in I\!R^+ \quad \text{and} \quad \Re\{\hat{F}_{33}^2\} > 0 \ . \qquad (29)$$

We verify this by considering the two cases separately (cf. also (28)). In addition, using $w^* = \frac{\sqrt{w}^*}{\sqrt{w}}$ if $|w| = 1$ we give the result of both normalizing operations (27) and (28) in terms of one single rotation/reflection and obtain the parameters (a, b, ε) by matching coefficients with (24)[1]:

Case A: $\epsilon = +1$:

$$\hat{F}_{33}^1 = \left|(G_R)_{33}^1\right| ww^* = \left|(G_R)_{33}^1\right| \in I\!R^+ \ ,$$

$$\hat{F}_{33}^2 = qw^{*2} \Rightarrow \operatorname{sgn}\left(\Re\{\hat{F}_{33}^2\}\right) = \epsilon = +1 \ ,$$

$$\hat{F}_{nl}(\zeta) = \left[\frac{(\sqrt{w}^* a'\zeta + \sqrt{w}b')(-\sqrt{w}^* b'^*\zeta + \sqrt{w}a')}{\zeta}\right]^l \cdot \left(\hat{F}_R\right)_{nl}\left(\frac{\sqrt{w}^* a'\zeta + \sqrt{w}b'}{-\sqrt{w}^* b'^*\zeta + \sqrt{w}a'}\right) \ ,$$

[1] Note that the symbol $\sqrt{w}$ is uniquely defined if we demand $\Re\{\sqrt{w}\} \geq 0$ and $\Im\{\sqrt{w}\} > 0$ if $\Re\{\sqrt{w}\} = 0$.

$$(a,b,\varepsilon) = (\sqrt{w}^* a', \sqrt{w} b', 1) \ .$$

Case B: $\epsilon = -1$:

$$\hat{F}_{33}^1 = (G_R)_{33}^{-1} w = \left|(G_R)_{33}^1\right| w^* w = \left|(G_R)_{33}^1\right| \in I\!R^+ \ ,$$

$$\hat{F}_{33}^2 = (G_R)_{33}^{-2} w^2 = -q^* w^2 = -(q w^{*2})^* \Rightarrow \operatorname{sgn}\left(\Re\{\hat{F}_{33}^2\}\right) = -\epsilon = +1 \ ,$$

$$\hat{F}_{nl}(\zeta) = \left[-\frac{(j\sqrt{w}b'\zeta + j\sqrt{w}^* a')(j\sqrt{w}a'\zeta - j\sqrt{w}^* b'^*)}{\zeta}\right]^l \cdot$$
$$\cdot \left(\hat{F}_R\right)_{nl} \left(\frac{j\sqrt{w}b'\zeta + j\sqrt{w}^* a'}{j\sqrt{w}a'\zeta - j\sqrt{w}^* b'^*}\right) \ ,$$

$$(a,b,\varepsilon) = (j\sqrt{w}b', j\sqrt{w}^* a', -1) \ .$$

Since it is not difficult to show that the identity is the only rotation/ reflection which preserves both normalizations (26) and (29) we conclude that we have derived a unique normalization procedure using exclusively moments of third order.

5 Complete moment invariants

In the previous section we computed uniquely all parameters of the orthogonal transformation which sends an object from the actual to its standard position. Now, to give the complete invariants, i.e. the moments of the object in its standard position we only have to return to (24). If we expand the expressions on both sides of this equation according to the definition of the ζ-coding of spherical moments (22) and match the coefficients of equal powers of ζ we obtain the following:

$$\hat{F}_{nl}^m = (\varepsilon a a^*)^l \left(\frac{a}{b}\right)^m \sum_{\mu=-l}^{l} \left(\hat{F}_R\right)_{nl}^{\mu} \left(-\varepsilon \frac{a}{b^*}\right)^{\mu} \sum_k \binom{l-\mu}{k-\mu} \binom{l+\mu}{k-m} \left(-\frac{bb^*}{aa^*}\right)^k \ .$$

Note that in the expression above no true divisions take place since it follows from $\max(\mu, m) \le k \le l + \min(0, m+\mu)$ that all exponents of a, a^*, b and b^* are nonnegative. With known parameters (a, b, ε) and measurements $\left(\hat{F}_R\right)_{nl}^{\mu}$ it is now clear that any desired invariant $\hat{F}_{nl}^m$ can be obtained.

6 Concluding remarks

In this article we have presented a unique pose determination procedure for orthogonal transformations of 3D objects based only on moments of order three. That means that moments of second order can still be used to normalize for an affine transformation as proposed in [2]. This is not the case if one performs normalization based on principal axes.

Furthermore, starting with geometrical moments we derived closed analytical expressions for all spherical moments of an object lying in a uniquely determined standard position. Since geometrical moments and therefore spherical moments too constitute a complete description of an object the presented invariants form a complete system.

We have not discussed computational complexity, but it is clear that since existing fast algorithms for 2D moments computation [7] can be readily generalized to 3D this problem will not be prohibitive for applications. Moreover, it can be shown that the fast moment generating algorithm described in [7] may be modified in such a way as to be able to compute the spherical moments directly from repeated cumulative sums on the object without the need to go through the geometrical moments. This results in increased numerical precision since the extreme dynamic range which is characteristic for geometrical moments is avoided. Finally, these considerations can be extended to 3D Zernike-like moments being orthonormal over a whole 3D region (e.g. the 3D unit sphere) and having improved classification power. These issues will be discussed elsewhere in due course.

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