Chapter 5c

Calculating invariants for discret objects

Invariants for Discrete Structures – An Extension of Haar Integrals over Transformation Groups to Dirac Delta Functions

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Summary

1. Introduction
2. Invariants for continuous objects
3. Invariants for discrete objects
   - Invariants for polygons
   - 3D-meshes
   - Discrimination performance and completeness
4. Experiments: Object classification in a Tangram database
5. Conclusions

Introduction

- Increased interest in 3D models and 3D sensors induce a growing need to support e.g. the automatic search in such databases
- As the description of 3D objects is not canonical ➔ use invariants for their description

Invariant integration over Euclidean group

For (cyclic) image translation and rotation:

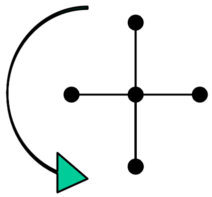
$$(g\mathbf{X})[i,j] = \mathbf{X}[k,l]$$

$$\begin{pmatrix} k \\ l \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} i \\ j \end{pmatrix} - \begin{pmatrix} t_0 \\ t_1 \end{pmatrix}$$

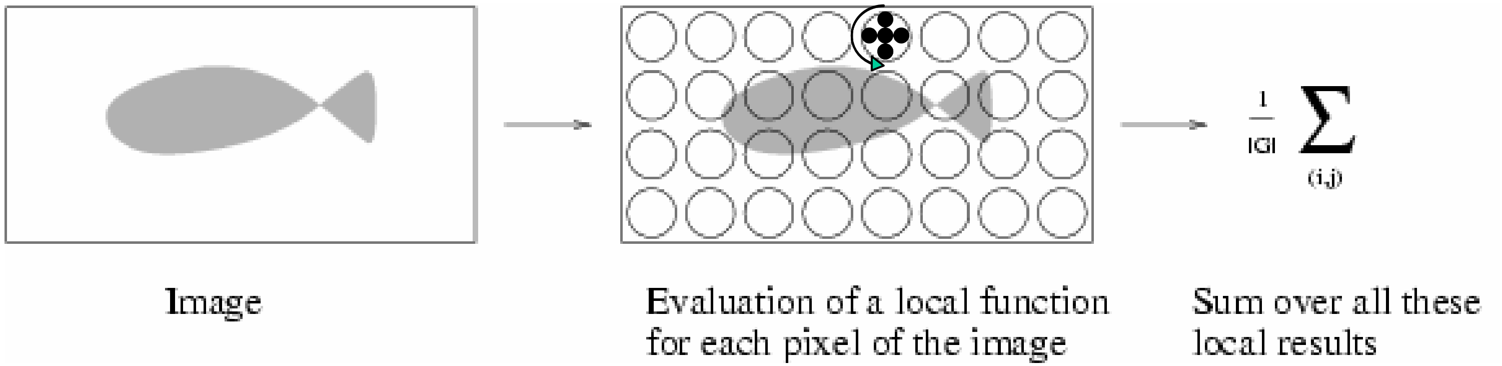
all indices to be understood modulo the image dimensions.

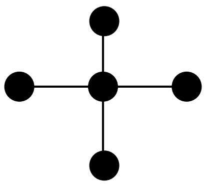
$$A[f](\mathbf{X}) = \frac{1}{2\pi NM} \int_{t_0=0}^{N} \int_{t_1=0}^{M} \int_{\varphi=0}^{2\pi} f(g\mathbf{X})\, d\varphi\, dt_1\, dt_0$$

Use as kernel functions monomials of pixels of local support and integrate over the Euclidean motion:

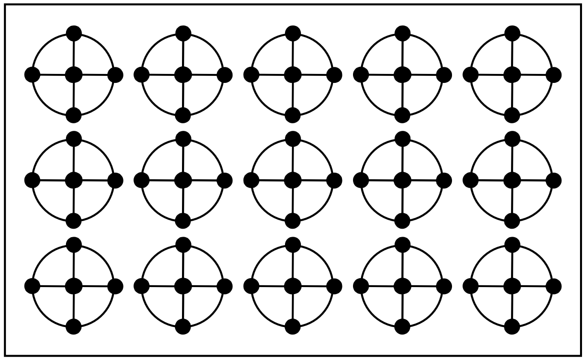


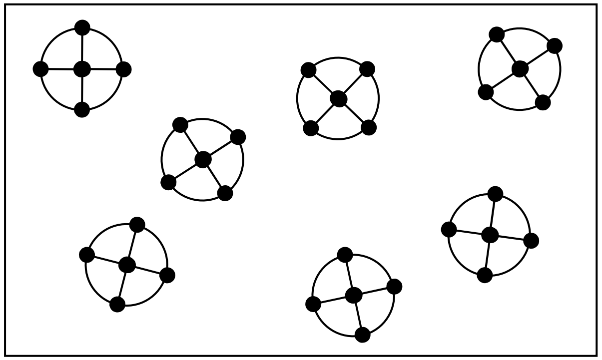
$$f(\mathbf{X}) = m_{00}^3 m_{01}^1 m_{0-1}^5 m_{10}^2 m_{-10}^3$$





Monomial

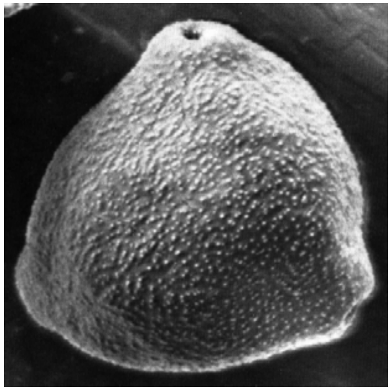




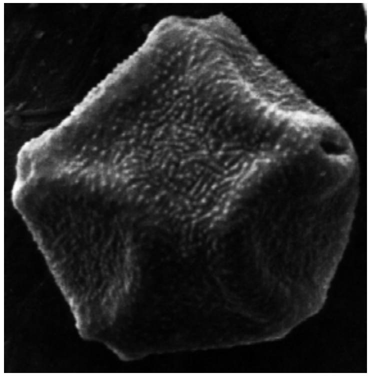
Deterministic integral over the planar Euclidean motion

Monte-Carlo-Integration

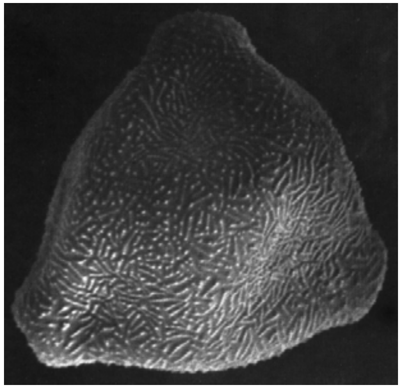
Pollen examples



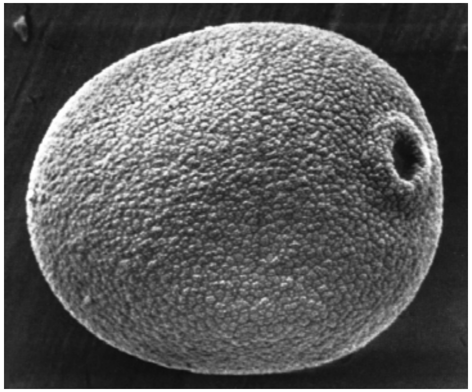
hazel



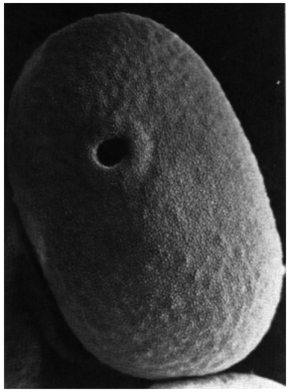
alder



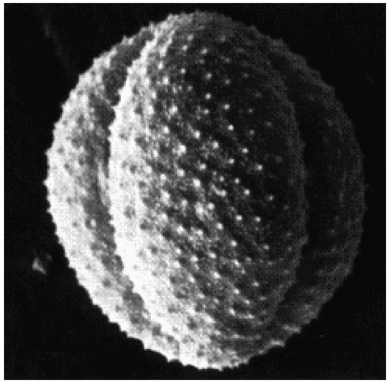
birch



grass



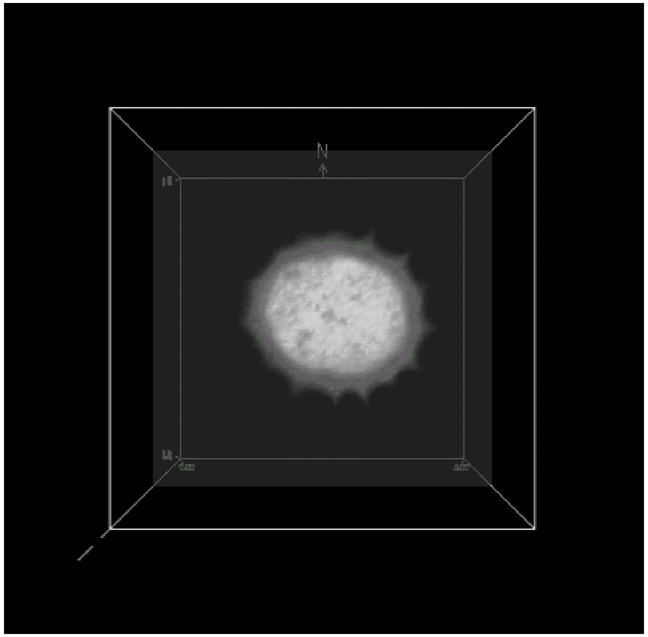
rye



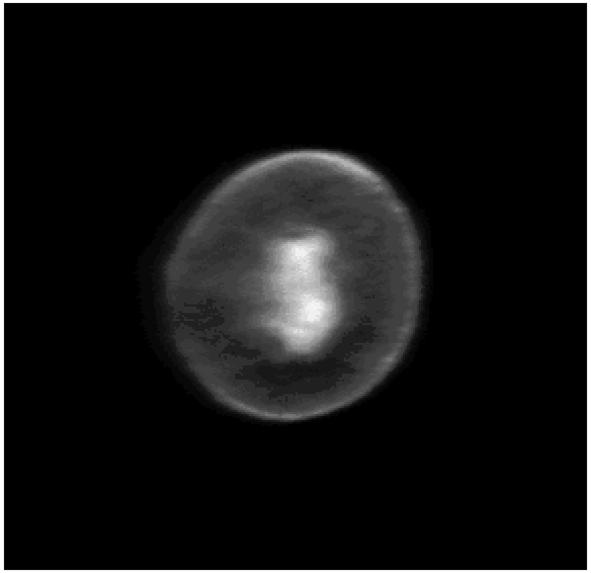
mugwort

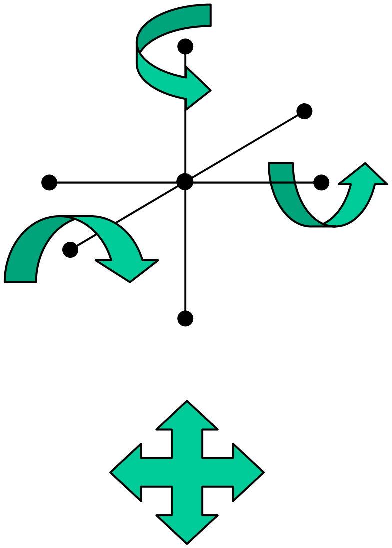
+ 33 further species (not relevant for allergies)

Daisy pollen grain



Taxus





Integrate over Euclidean Motion

Classification Results using 3D LSM Data

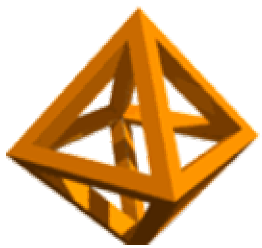
(leave-one-out Classification)

	Correct	Wrong classifications
Artemisia:	13	1 -> *Compositae*, 1 -> *Platanus*
Alnus:	15	-
Alnus viridis:	12	-
Betula:	13	2 -> *Plantago*
Corylus:	13	1 -> *Alnus*
Gramineae/Poaceae:	15	-
Secale:	11	3 -> *Fagus*, 1 -> *Tilia*
Allergolocial irrelevant*:	282	2 -> *Gramineae*
Total:	97.4%	2.6%

*** *Acer, Carpinus, Chenopodium, Compositae, Cruciferae, Fagus, Quercus, Aesculus, Juglans, Fraxinus, Plantago, Platanus, Rumex, Populus, Salix, Taxus, Tilia, Ulmus, Urtica*

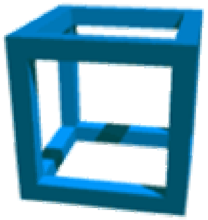
The Five Platonic Solids

Octahedron

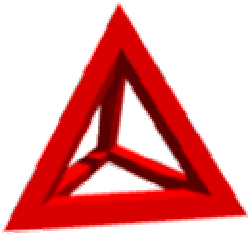


Icosahedron.gif

Dodecahedron



Hexahedron



Tetrahedron

A platonic solid is a polyhedron all of whose faces are congruent regular polygons, and where the same number of faces meet at every vertex. The best know example is a *cube* (or *hexahedron*) whose faces are six congruent squares.

Extension of Haar-Integrals to Discrete Structures

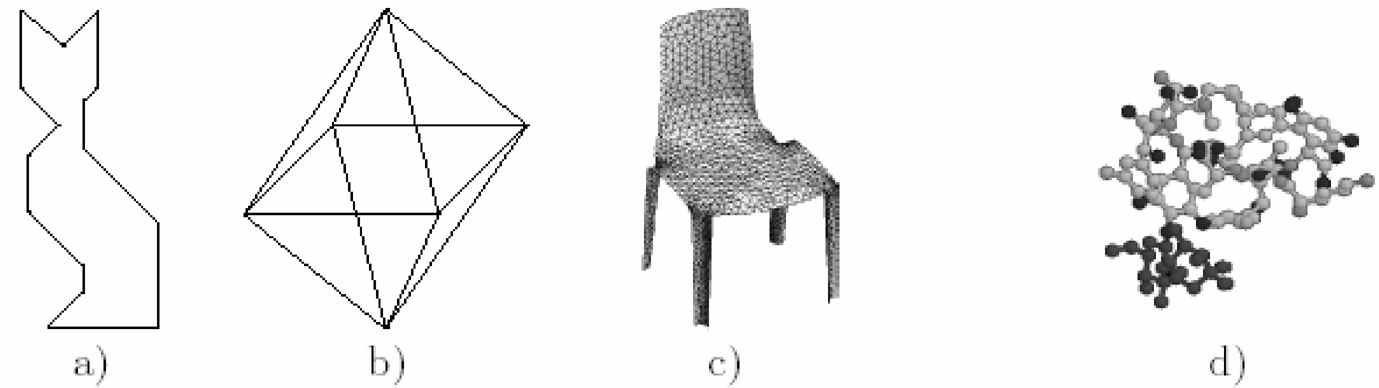


Fig. 1. Discrete structures in 2D and 3D: (a) closed contour described by a polygon (b) wireframe object (c) 3D triangulated surface mesh (d) molecule.

Describe discrete structures with Dirac delta functions!

Invariants for discrete objects

1. For a discrete object Δ and a kernel function $f(\Delta)$ it is possible to construct an invariant feature $T[f](\Delta)$ by integrating $f(g\Delta)$ over the transformation group $g \in G$.
2. The kernel function is properly designed, such that it delivers a value dependent on the discrete features of a local neighborhood, when a vertex of the object moved by the continuous Euclidean motion g hits the origin and has one specific orientation.
3. Let us assume that our discrete object is different from zero only at its vertices. A rotation and translation invariant local discrete kernel function h takes care for the algebraic relations to the neighboring vertices and we can write:

$$f\left(\Delta\right) = \sum_{i \in \mathbb{V}} h(\Delta, \mathbf{x}_i)\,\delta(\mathbf{x} - \mathbf{x}_i)$$

where $\mathbb{V}$ is the set of vertices and $\mathbf{x}_i$ the vector representing vertex i.

4. In order to get finite values from the distributions it is necessary to introduce under the Haar integral another integration over the spatial domain $\mathbf{X}$.
5. By choosing an arbitrary integration path in the continuous group G we can visit each vertex in an arbitrary order the integral is transformed into a sum over all local discrete functions allowing all possible permutations of the contributions of the vertices.

Extension of Haar-Integrals to Discrete Structures

$$T[f](\Delta) := \int_G \int_{\mathbf{X}} f(g\Delta) d\mathbf{x} dg = \int_G \left[\int_{\mathbf{X}} \sum_{i \in \mathbb{V}} h(g\Delta, g\mathbf{x}_i) \delta(g\mathbf{x} - g\mathbf{x}_i) d\mathbf{x} \right] dg$$

$$= \int_G \left[\sum_{i \in \mathbb{V}} h(\Delta, \mathbf{x}_i) \right] dg = \sum_{i \in \mathbb{V}} h(\Delta, \mathbf{x}_i)$$

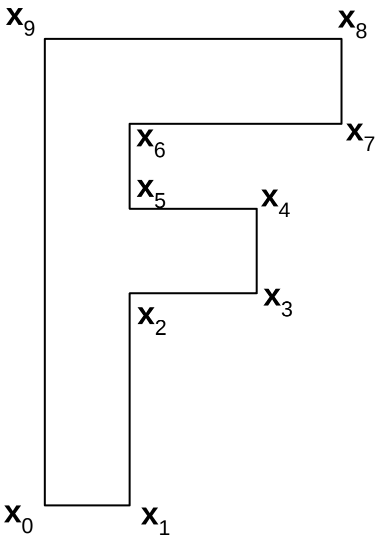
Intuitive result: get *global* Euclidean invariants by summation over discrete *local* Euclidean invariants $h(\Delta, \mathbf{x}_i)$!

Remember: The delta function has the following selection property:

$$\int_{-\infty}^{+\infty} f(x) \delta(x - a) dx = f(a)$$

Euclidean Invariants for Polygons

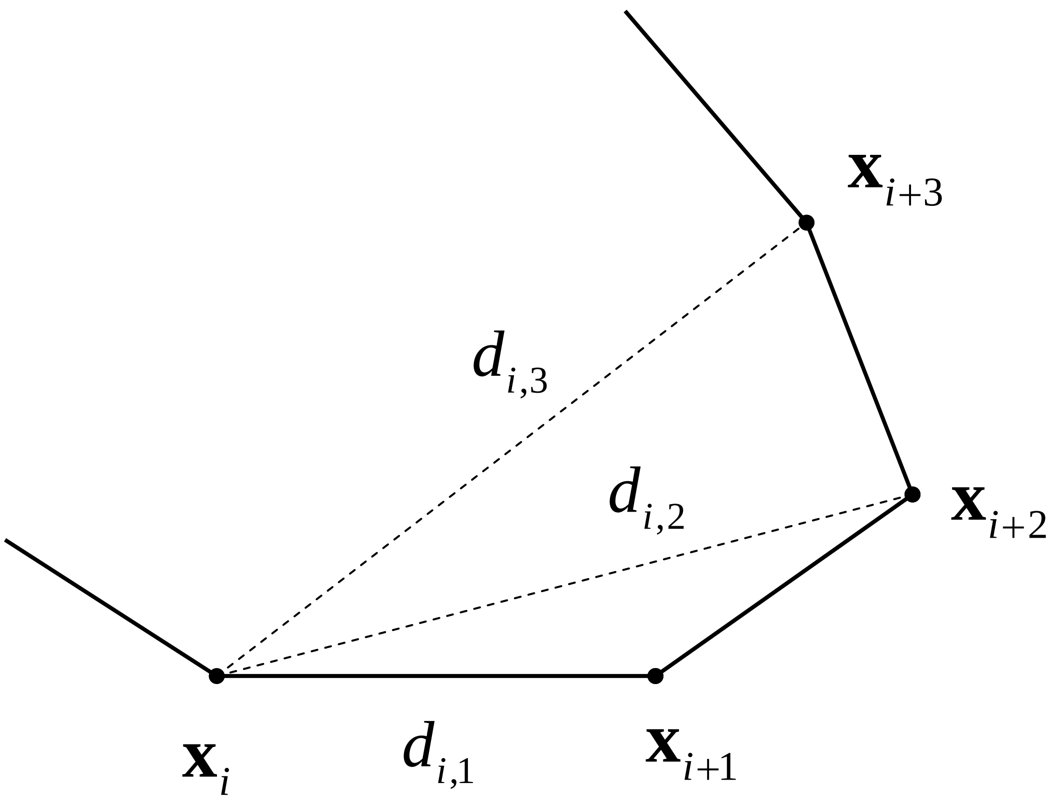
We assume e.g. to have given a polygon with 10 vertices, e.g.



$$\{\mathbf{x}_i\} = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} 0 & 1 & 1 & 2.5 & 2.5 & 1 & 1 & 3.5 & 3.5 & 0 \\ 0 & 0 & 2.5 & 2.5 & 3.5 & 3.5 & 4.5 & 4.5 & 5.5 & 5.5 \end{array}\right]$$

Choose as local Euclidean invariants distances of vertex i and its k-th righthand neighbours:

$$d_{i,k} = \left\| \mathbf{x}_i - \mathbf{x}_{<i+k>} \right\|$$

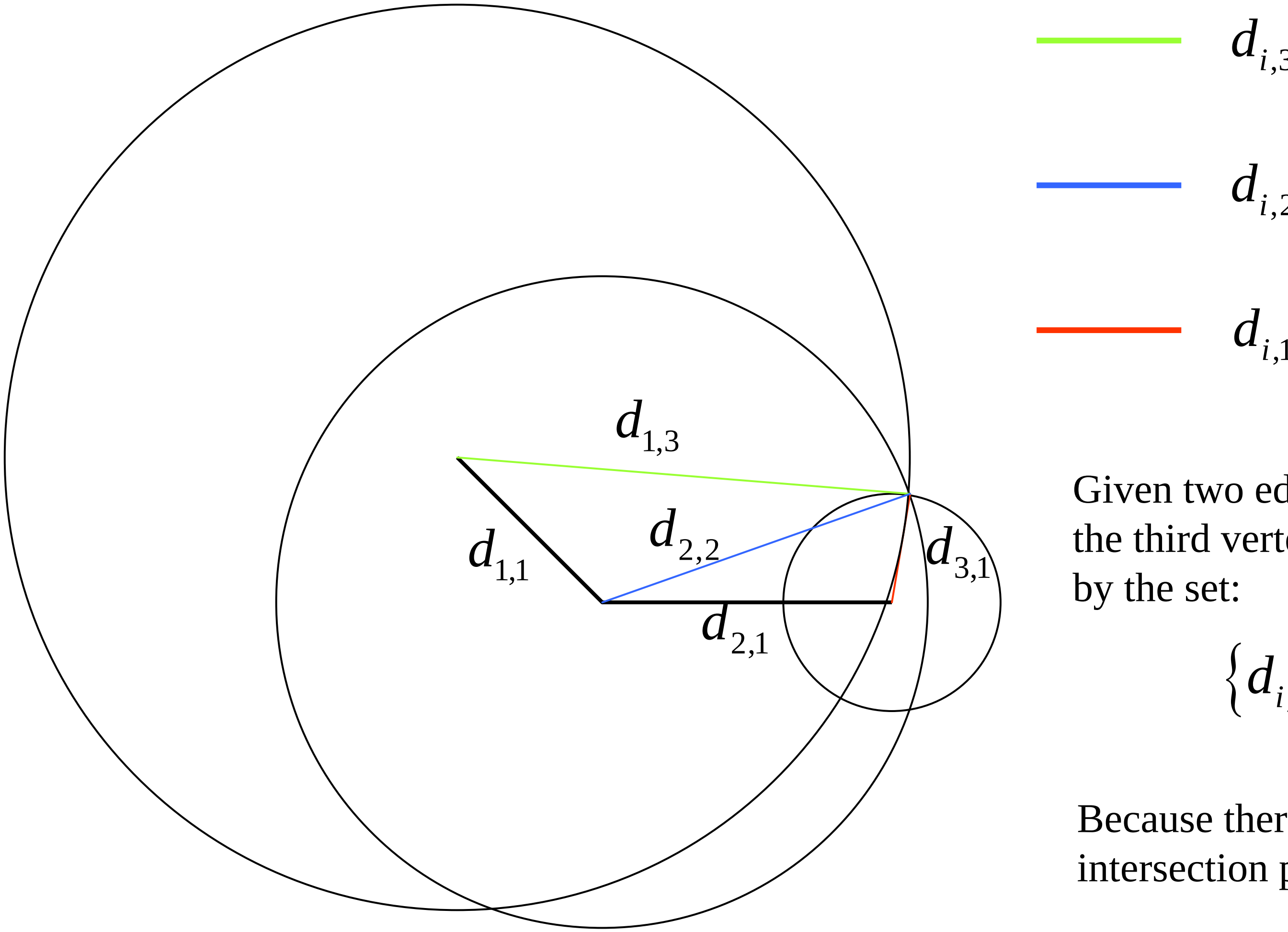


The elements:

$$\left\{ d_{i,1}, d_{i,2}, d_{i,3} \right\}$$

form a basis for a polygon, because they uniquely define a polygon (up to a mirror-polygon) !

Principle of rigidity



Given two edges $d_{1,1}$ and $d_{2,1}$. Then the third vertex is uniquely defined by the set:

$$\left\{d_{i,1}, d_{i,2}, d_{i,3}\right\}$$

Because there is a unique intersection point of three circles.

This means, that a whole polygon can be uniquely generated by this basis elements iteratively.

As discrete functions of local support we derive *monomials* from distances between neighbouring vertices and hence we get invariants by summing these discrete functions of local support (DFLS) over all vertices:

$$\tilde{x}_{n_1,n_2,n_3,n_4} = \sum_{i \in \mathbb{V}} h(\Delta, \mathbf{x}_i) = \sum_{i \in \mathbb{V}} d_{i,1}^{n_1} d_{i,2}^{n_2} d_{i,3}^{n_3} d_{i,4}^{n_4}$$

Chosing the following 8 values for the exponents we would end up with a corresponding invariant feature vector and a set of 8 invariants:

i	n_1	n_2	n_3
$\tilde{x}_0$	1	0	0
$\tilde{x}_1$	1	1	0
$\tilde{x}_2$	1	0	1
$\tilde{x}_3$	1	1	1
$\tilde{x}_4$	2	0	0
$\tilde{x}_5$	2	1	0
$\tilde{x}_6$	2	0	1
$\tilde{x}_7$	2	1	1

We clearly recognize e.g. $\tilde{x}_0$

as the circumference of the polygon as an invariant.
For the above example of the letter F we get the following invariants:

$$\tilde{\mathbf{x}} = \begin{bmatrix} 21 & 44 & 83.82 & 68.6 & 184.6 & 665.3 & 149.5 & 751.6 \end{bmatrix}^T$$

How complete is this set of invariants?

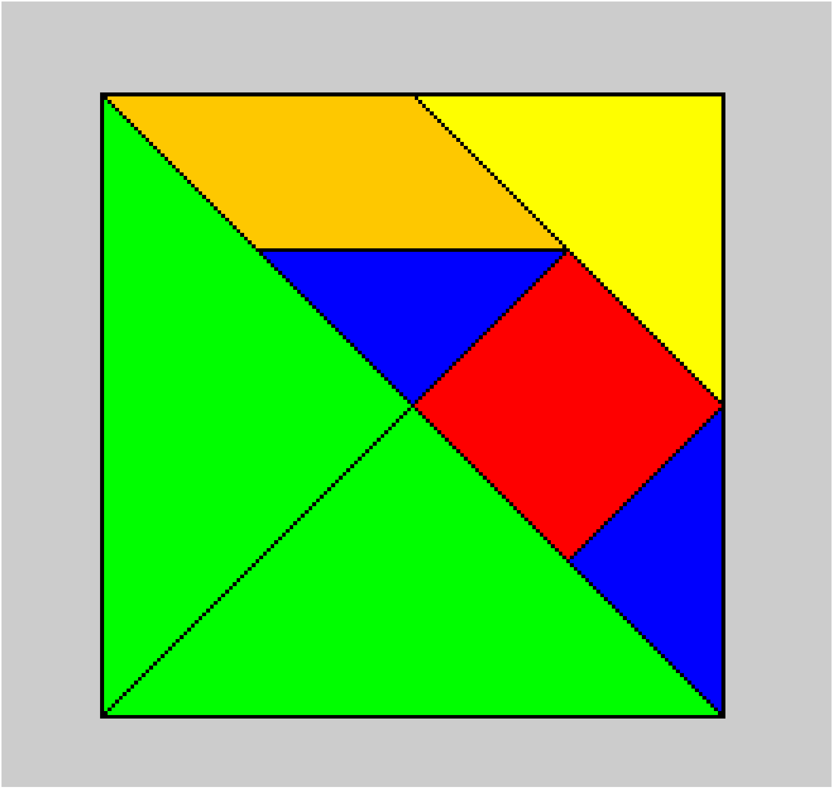
Completeness for finite Groups

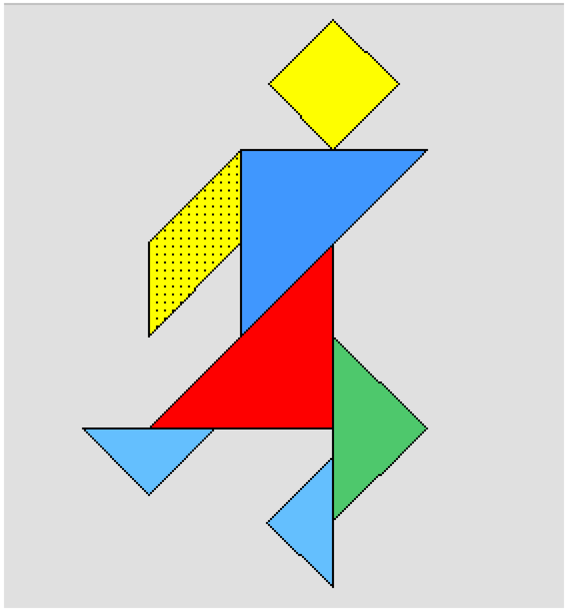
(Emmy Noether, 1916)

For finite Groups G with $|G|$ elements and patterns of dimensionality N the group averages over all monomials of degree $\leq |G|$ are complete and form a basis of the pattern space. The number of monomials is given by

$$\binom{N+|G|}{N}$$

Experiment: Object classification in a Tangram database

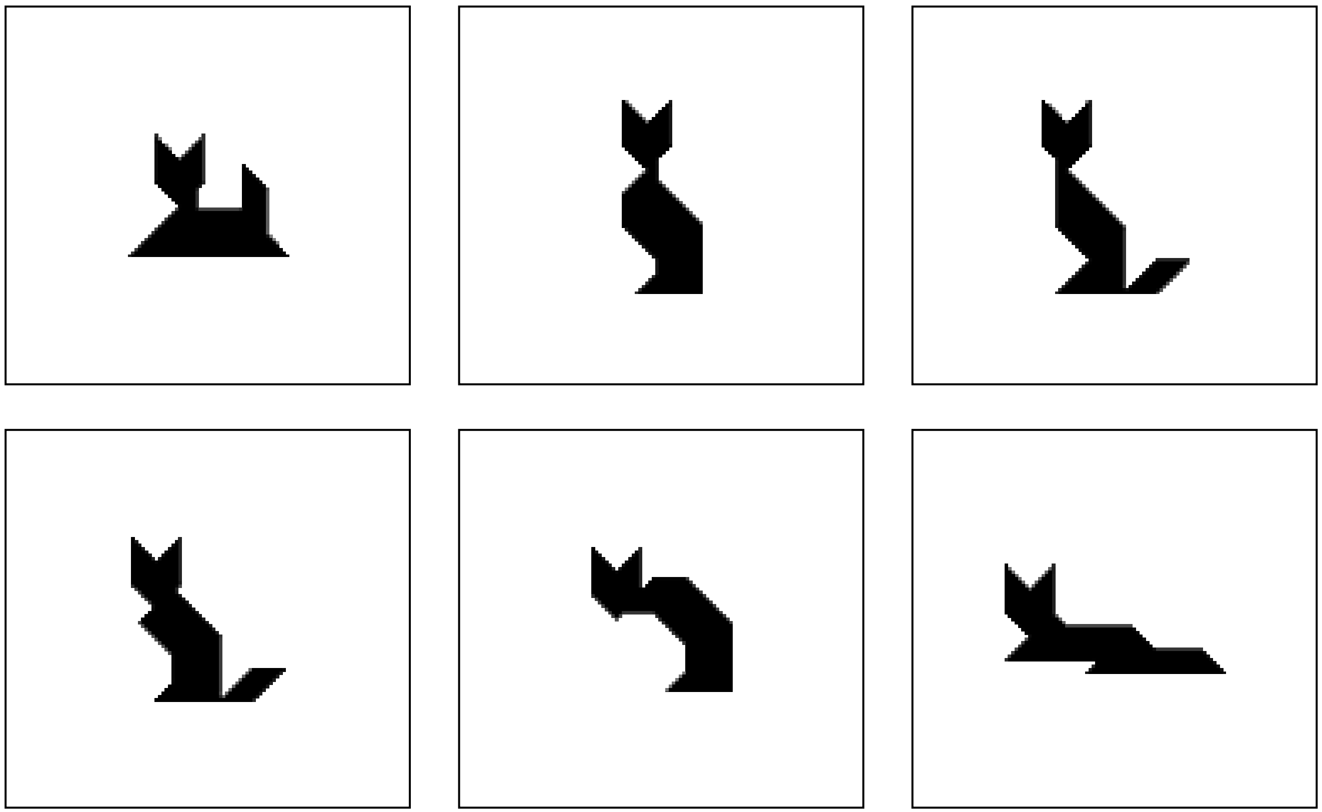


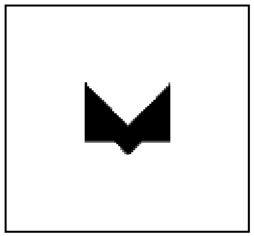


Take the outer contour as feature

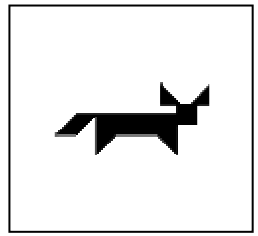
Objects can not easily be discriminated with trivial geometric features!

Cats





bat.png
n1029923508



cow.png
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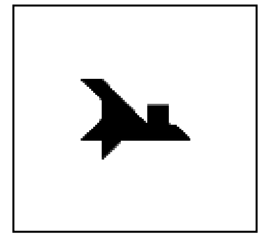
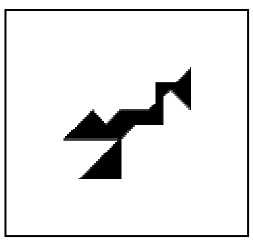
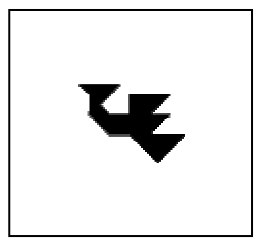


seal.png
n1033206320



crab1.png

n1029696706



crab2.png

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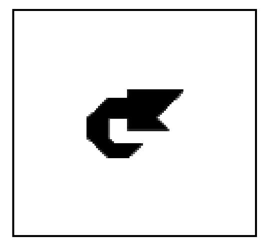

crab3.png

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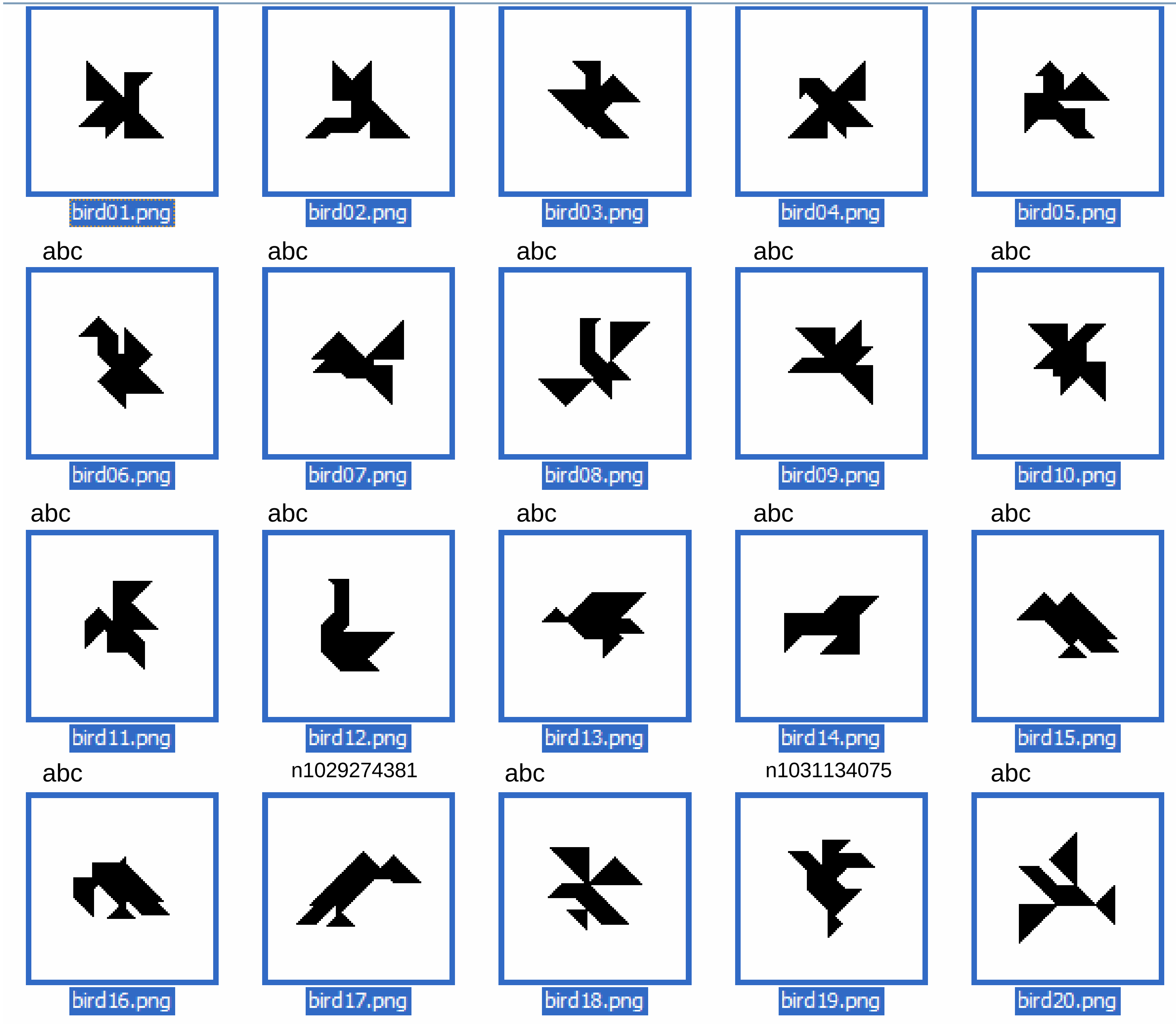
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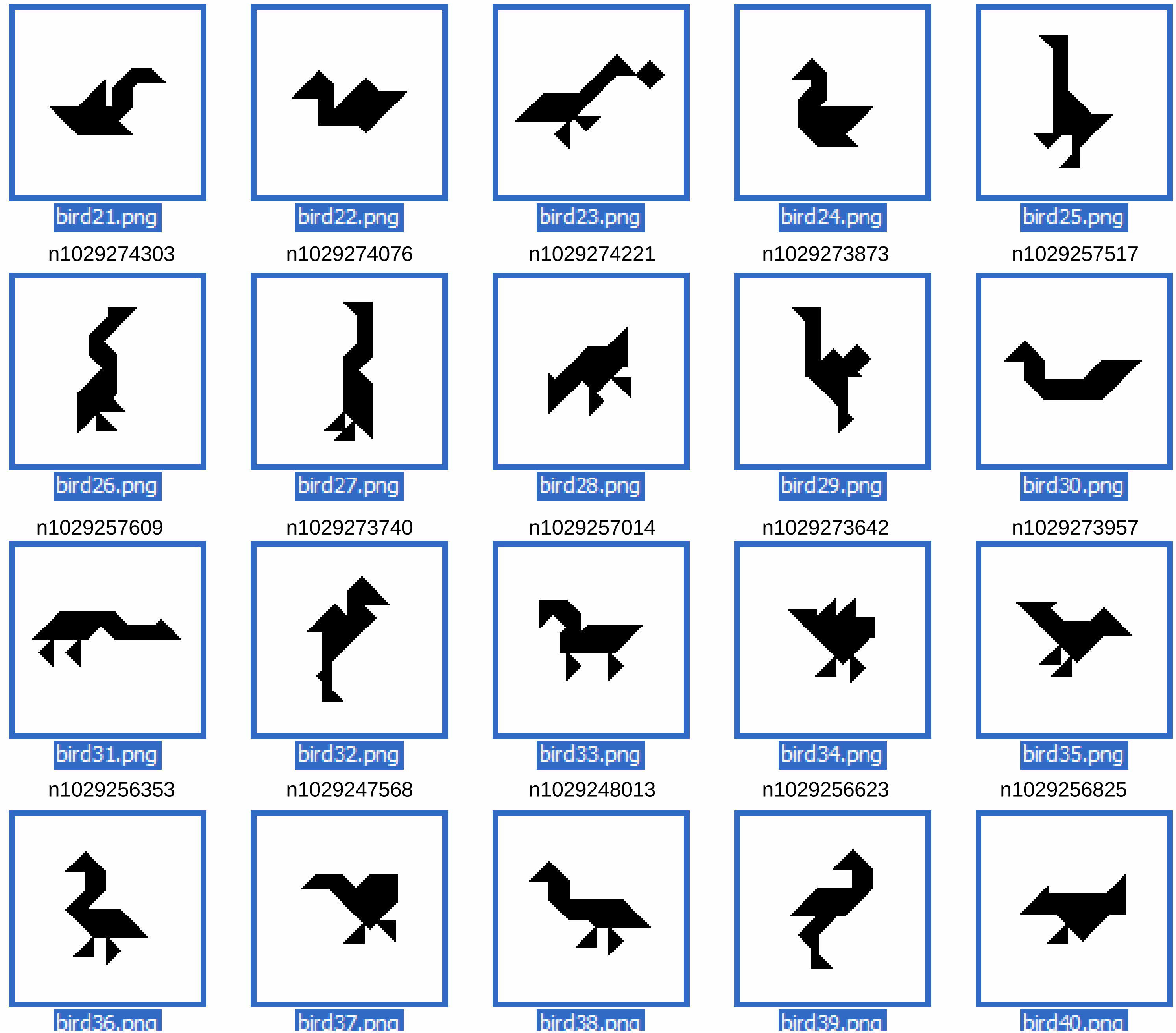
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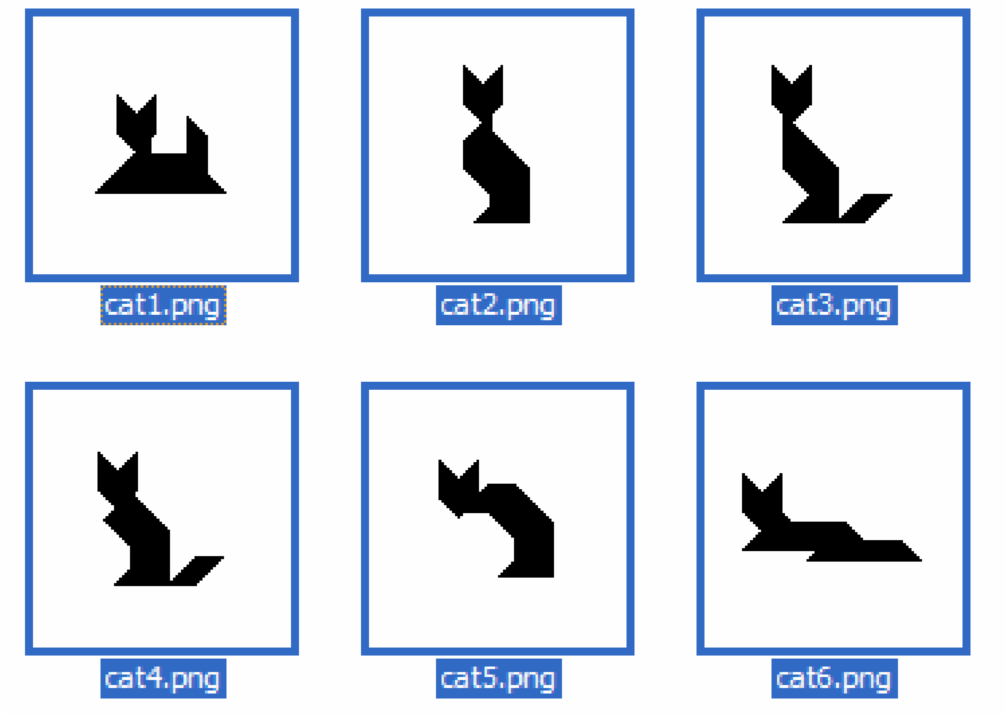

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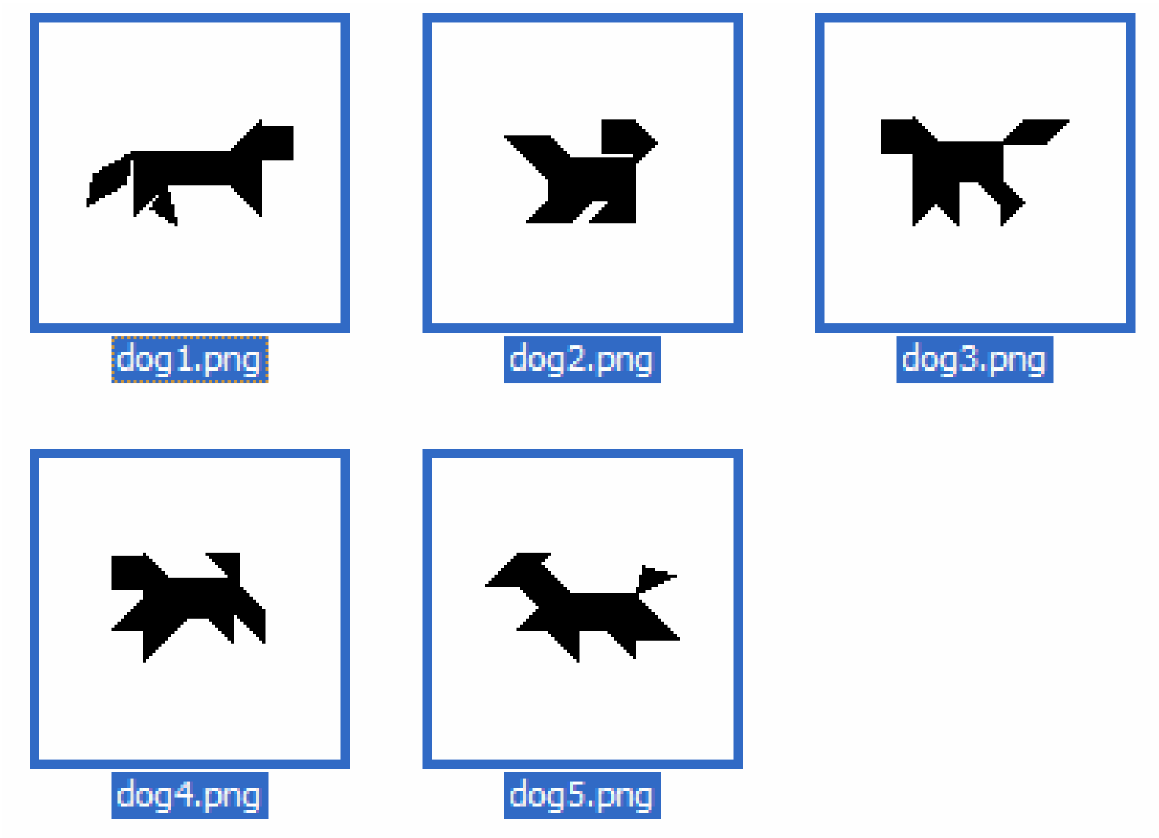

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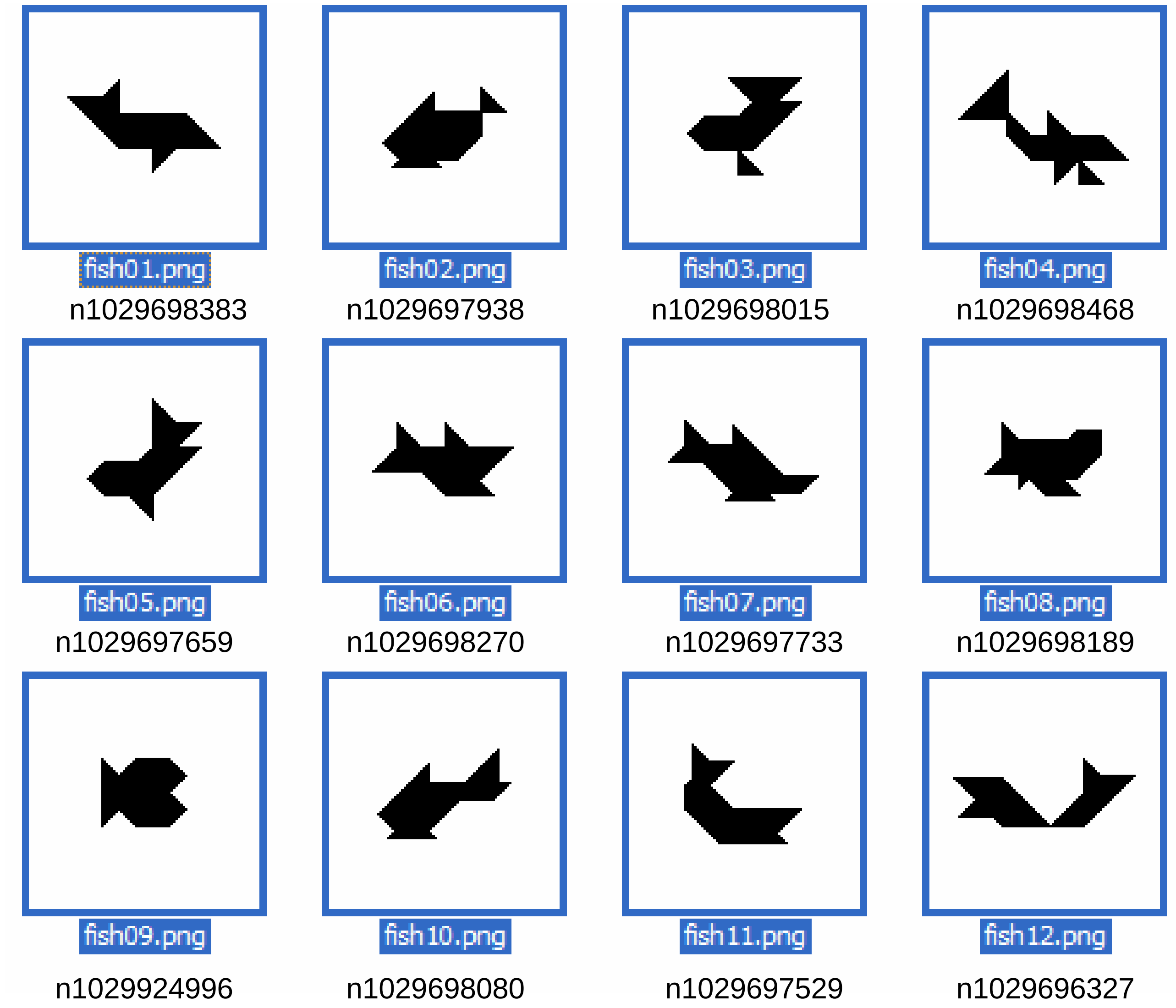
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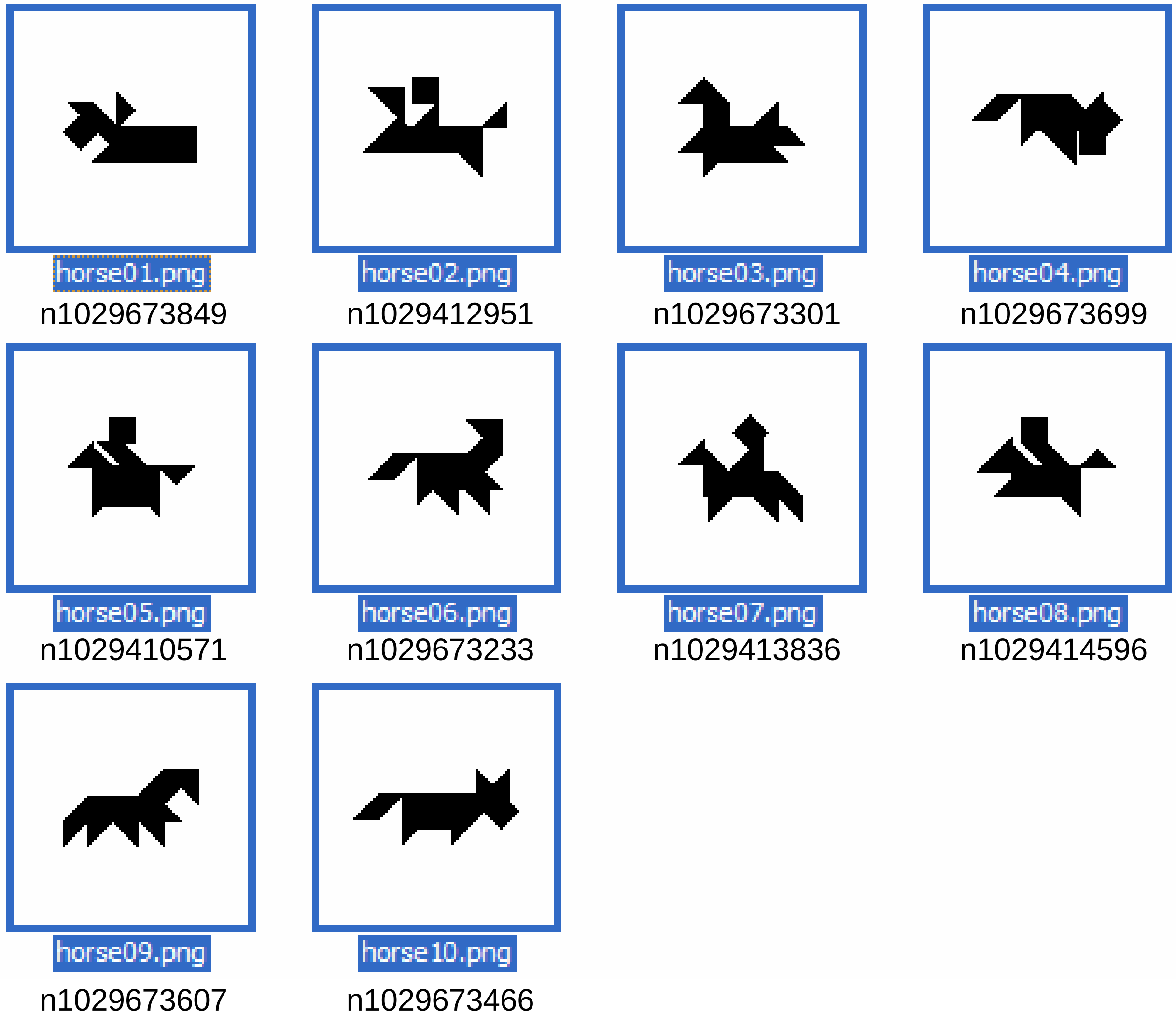
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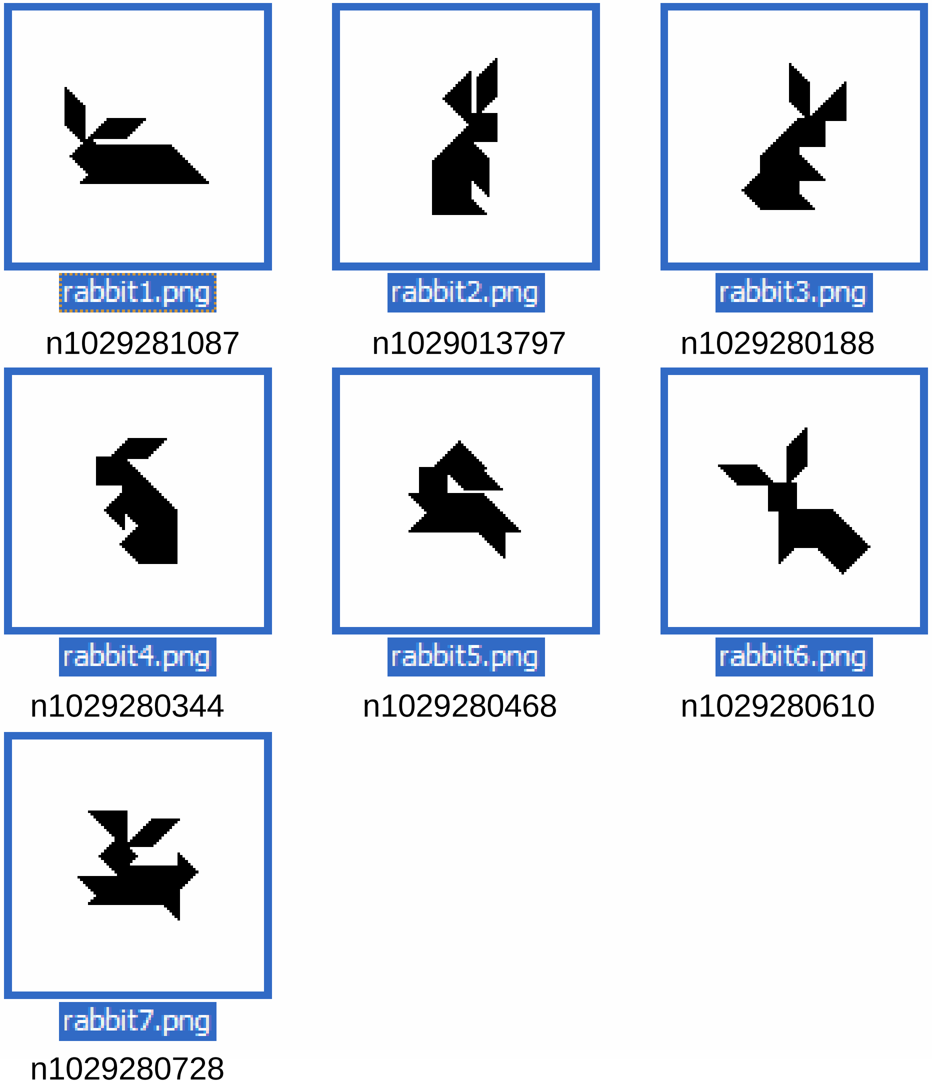
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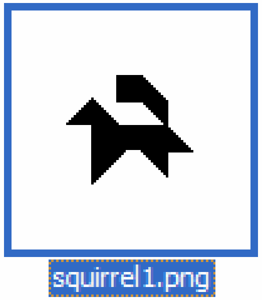

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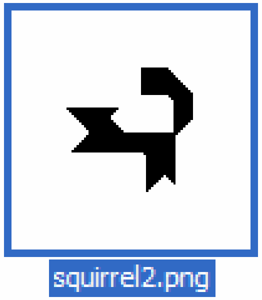

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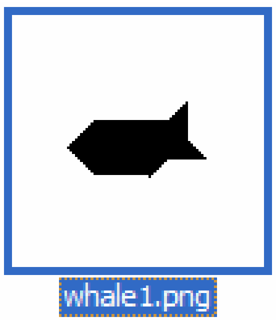

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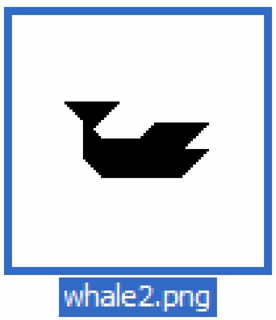

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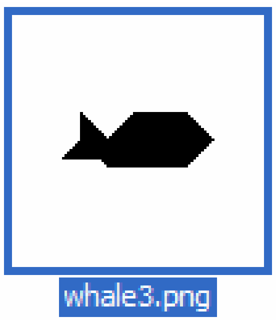

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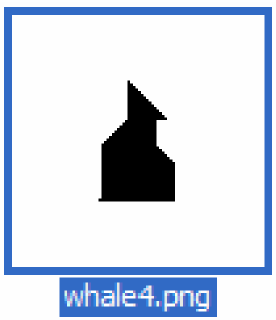

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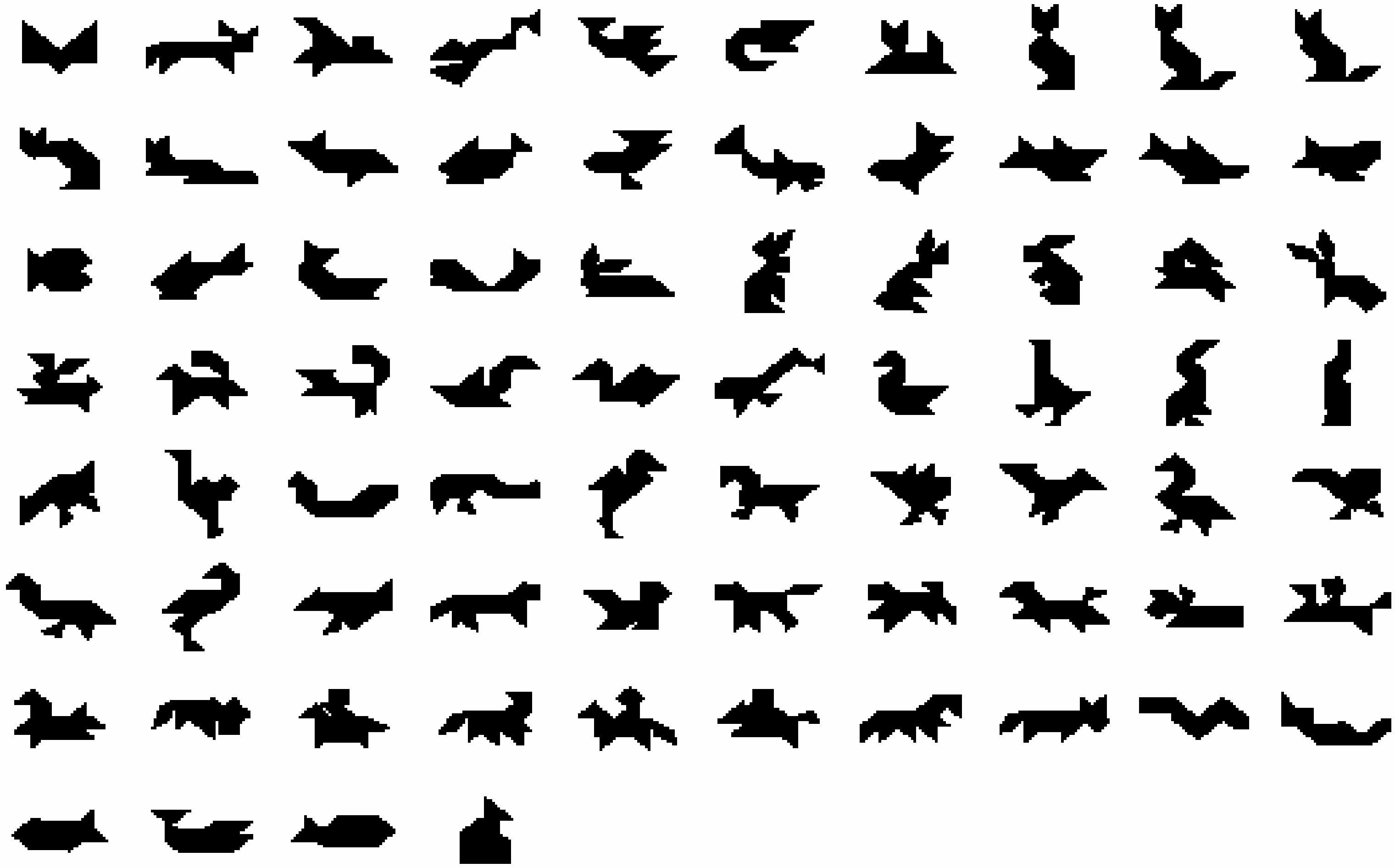


Fig. 6. The 74 tangrams used in the experiment.

The experiments were conducted with three sets of 6, 10 and 14 invariants respectively using the following exponent table:

i	$\tilde{x}_0$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\tilde{x}_5$	$\tilde{x}_6$	$\tilde{x}_7$	$\tilde{x}_8$	$\tilde{x}_9$	$\tilde{x}_{10}$	$\tilde{x}_{11}$	$\tilde{x}_{12}$	$\tilde{x}_{13}$
n_1	1	0	0	1	1	0	0	1	0	0	2	0	0	0
n_2	0	1	0	1	0	1	0	0	1	0	0	2	0	0
n_3	0	0	1	0	1	1	0	0	0	1	0	0	2	0
n_4	0	0	0	0	0	0	1	1	1	1	0	0	0	2

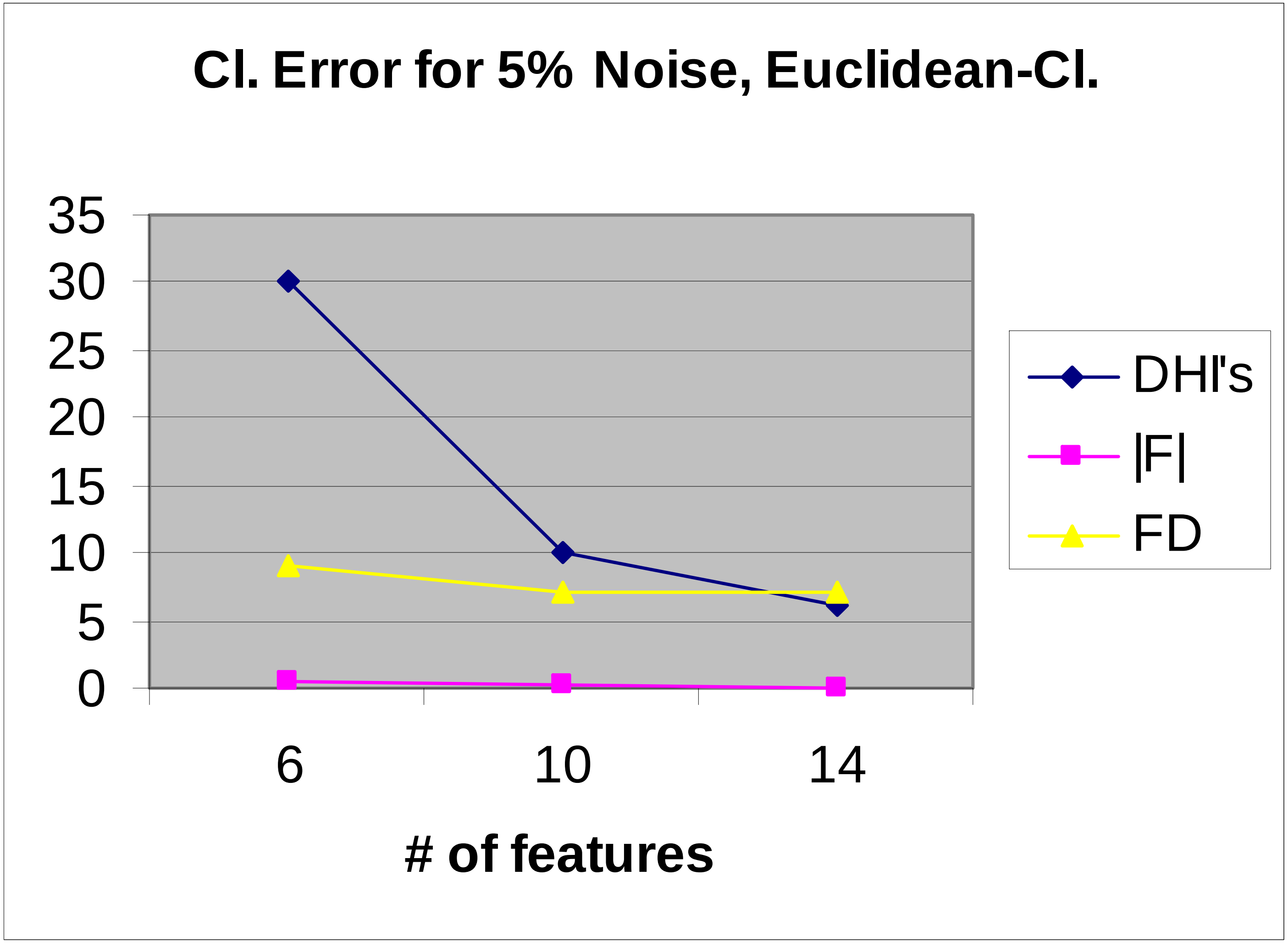
$$\tilde{x}_{n_1,n_2,n_3,n_4} = \sum_{i\in\mathbb{V}} h(\Delta, \mathbf{x}_i) = \sum_{i\in\mathbb{V}} d_{i,1}^{n_1} d_{i,2}^{n_2} d_{i,3}^{n_3} d_{i,4}^{n_4}$$

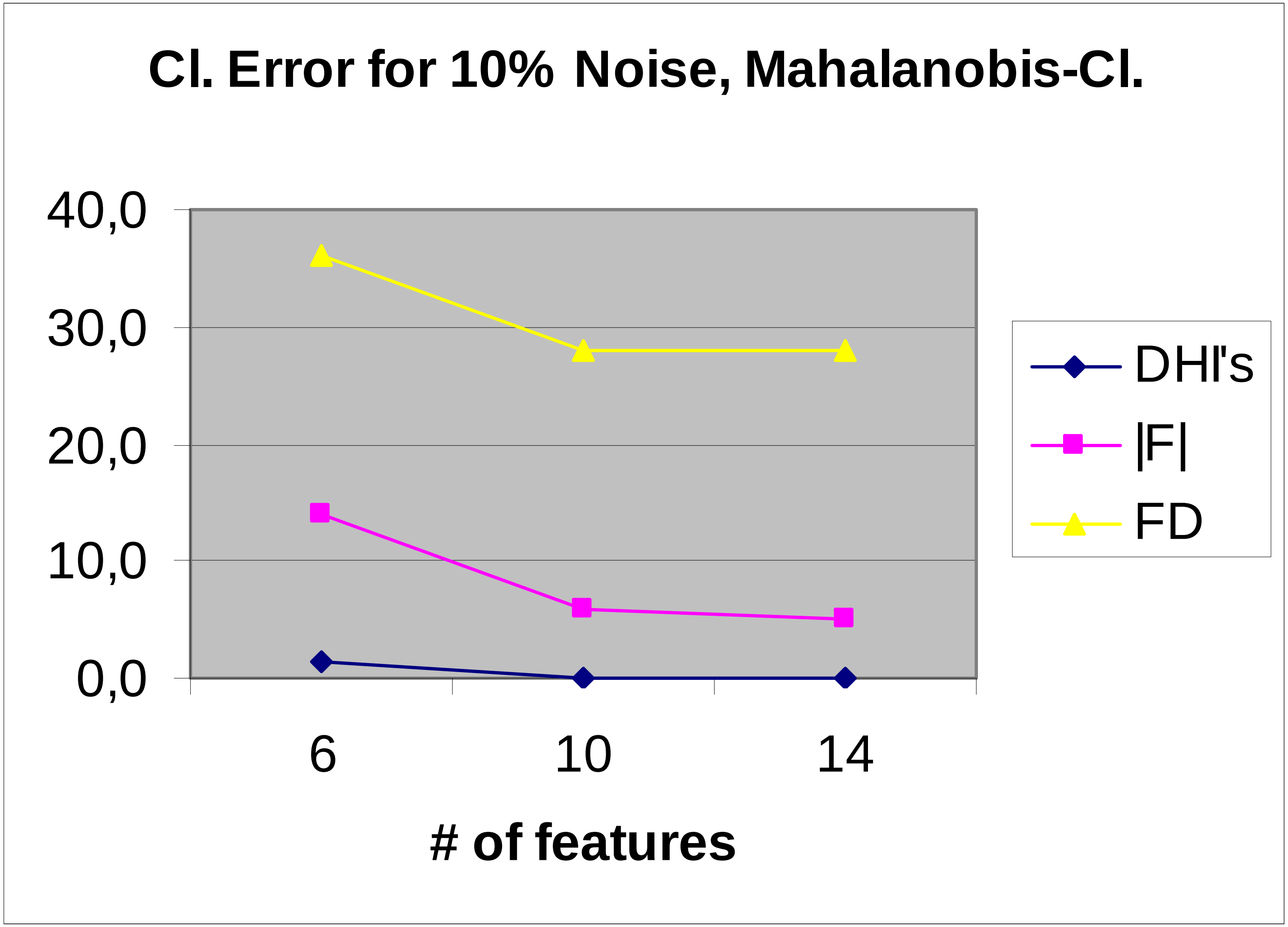
The classification performance was measured against additive noise of 5%, 10% and 20%.

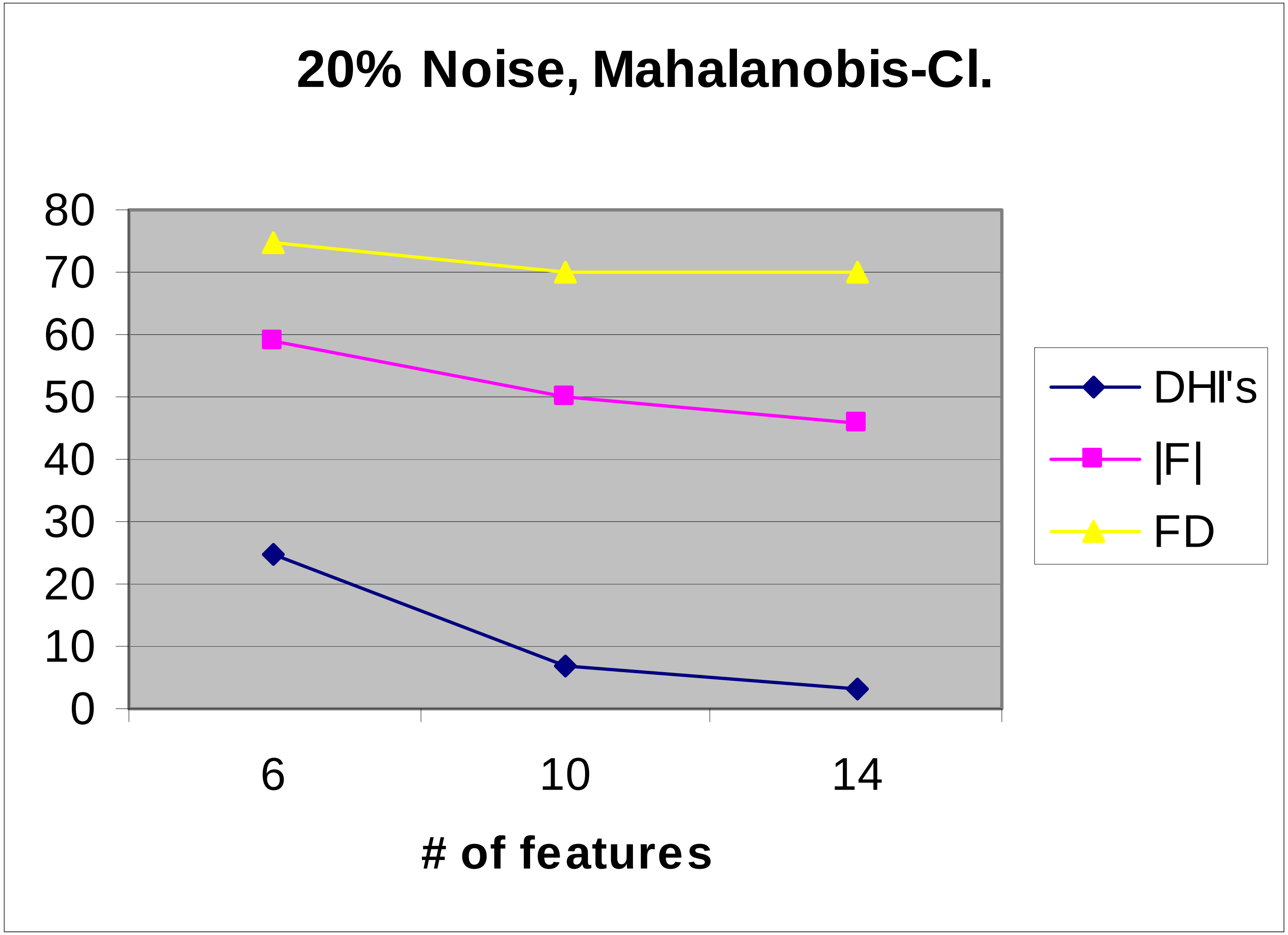
Classification error (in percent) for 5%, 10% and 20% noise for 74 tangrams with a Euclidean (E) and a Mahalanobis (M) Classifier

noise (in percent)	# of invariants	metric	DHI‘s class. Error in %	FD	|F|
5	6	E	30	9	0.5
5	10	E	10	7	0.2
5	14	E	6	7	0
10	6	M	1.5	36	14
10	10	M	0	28	6
10	14	M	0	28	5
20	6	M	25	75	59
20	10	M	7	70	50
20	14	M	3	70	46

Empirical evaluation for the degree of completeness!

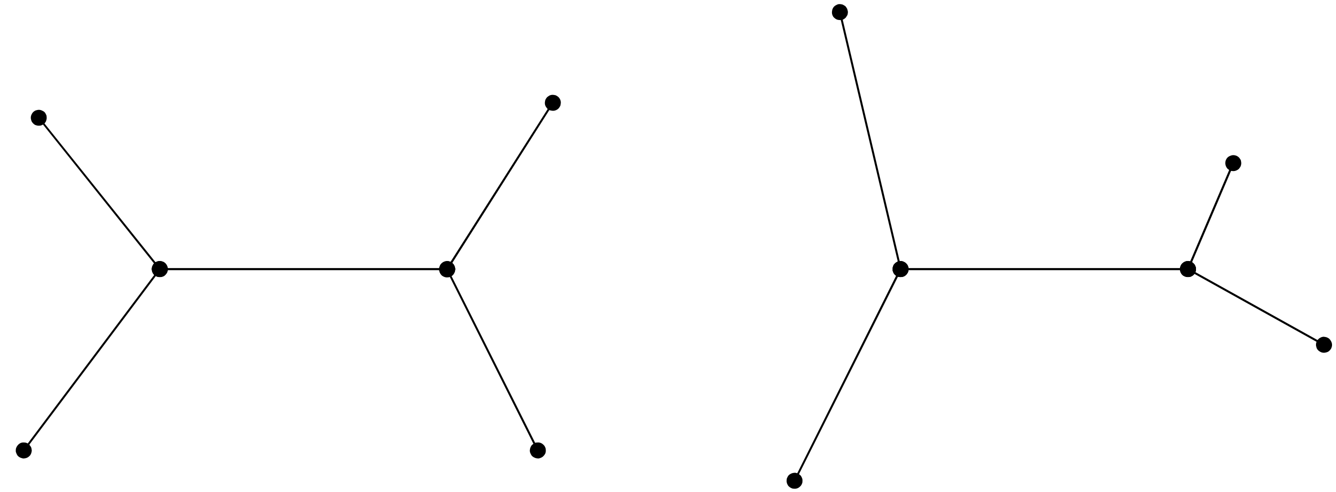

Cl. Error for 5% Noise, Euclidean-Cl.
35
30
25
20
15
10
5
0
6
10
14
# of features
DHI's
|F|
FD


Cl. Error for 10% Noise, Mahalanobis-Cl.
40,0
30,0
20,0
10,0
0,0
6
10
14
# of features
DHI's
|F|
FD

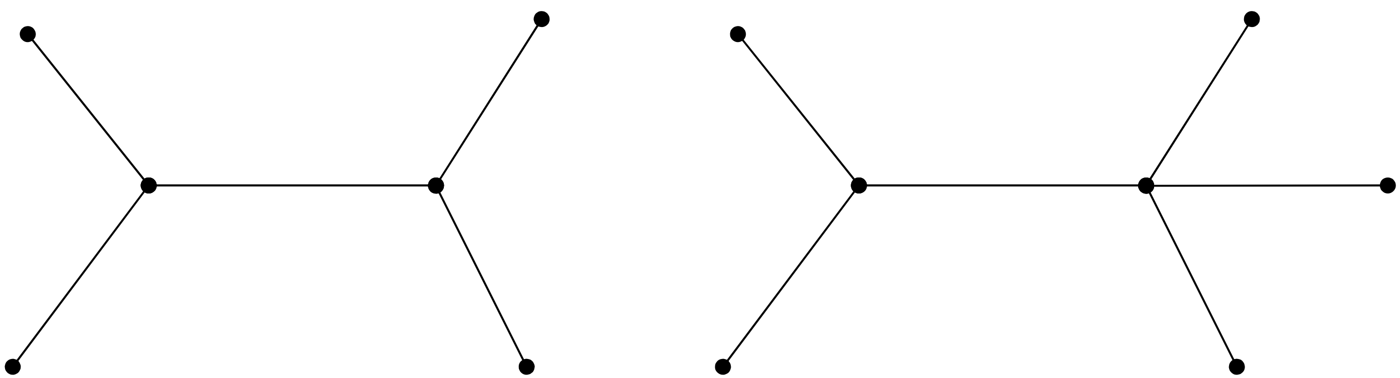

20% Noise, Mahalanobis-Cl.
80
70
60
50
40
30
20
10
0
6
10
14
# of features
DHI's
|F|
FD

Topologically equivalent structures and (TE) non topologically equivalent structures (NTE)

TE



NTE



Properties

- If we constrain our calculation to a finite number of invariants we end up with a simple *linear complexity* in the number of vertices. This holds if the local neighborhood of vertices is resolved already by the given data structure; otherwise the cost for resolving local neighborhoods must be added.
- In contrast to *graph matching* algorithms we apply here algebraic techniques to solve the problem. This has the advantage that we can apply *hierarchical searches* for retrieval tasks, namely, to start only with one feature and hopefully eliminate already a large number of objects and then continue with an increasing number of features etc.

Conclusion

- In this paper we have introduced a novel set of invariants for discrete structures in 2D and 3D.
- The construction is a rigorous extension of Haar integrals over transformation groups to Dirac Delta Functions.
- The resulting invariants can easily be calculated with linear complexity in the number of vertices.
- The proposed approach has the potential to be extended to other discrete structures and even to the more general case of weighted graphs.

Literature: (http://lmb.informatik.uni-freiburg.de)

(1) S. Siggelkow and H. Burkhardt. Image retrieval based on local invariant features. In Proceedings of the IASTED International Conference on Signal and Image Processing (SIP) 1998, pages 369-373, Las Vegas, Nevada, USA, October 1998. IASTED.

(2) M. Schael and H. Burkhardt. Automatic detection of errors on textures using invariant grey scale features and polynomial classifiers. In M. K. Pietikäinen, editor, Texture Analysis in Machine Vision, volume 40 of Machine Perception and Artificial Intelligence, pages 219-230. World Scientific, 2000.

(3) O. Ronneberger, U. Heimann, E. Schultz, V. Dietze, H. Burkhardt and R. Gehrig. Automated pollen recognition using gray scale invariants on 3D volume image data. Second European Symposium on Aerobiology, Vienna/Austria, Sept. 5-9, 2000.

(4) H. Burkhardt and S. Siggelkow. Invariant features in pattern recognition - fundamentals and applications. In C. Kotropoulos and I. Pitas, editors, Nonlinear Model-Based Image/Video Processing and Analysis, pages 269-307. John Wiley & Sons, 2001.