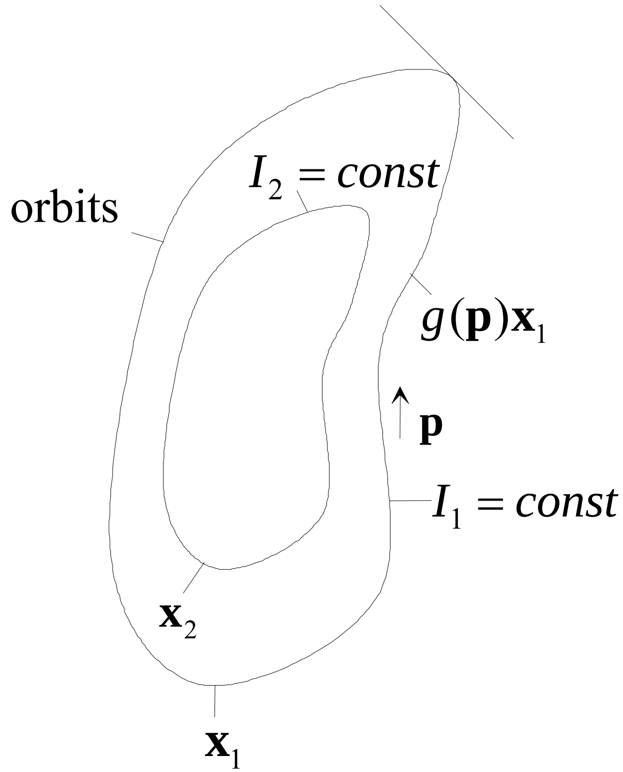
Chapter 5

General approaches for computing invariants

The three canonical possibilities for computing invariants (necessary constraints)

1) <u>integration over transformation group</u>
(Haar integral, Hurwitz 1897)



$$I = \int_{\mathcal{G}} f(g(\mathbf{p})\mathbf{x})dg$$

e.g. f : polynoms

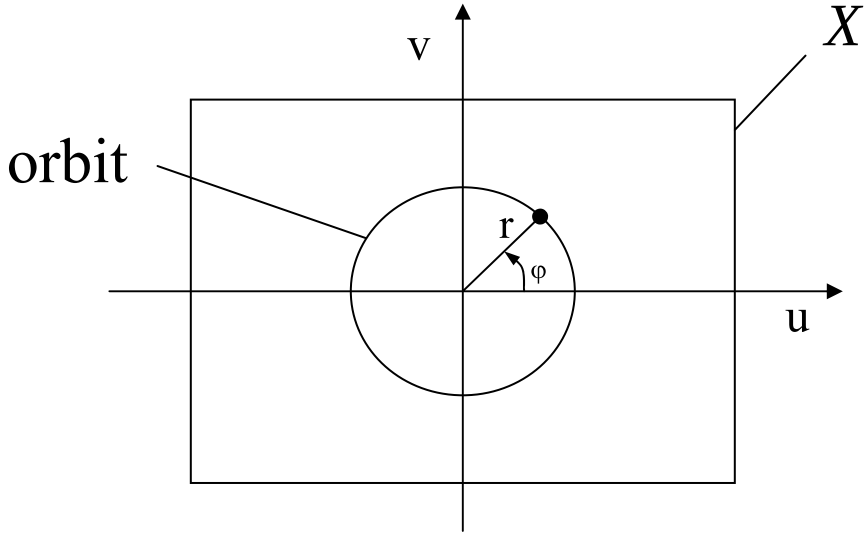
2) <u>differential approach</u>

$$\frac{\partial I(g(\mathbf{p})\mathbf{x})}{\partial p_i} \equiv 0 \qquad \text{for arbitrary } \mathbf{p}$$

solve according partial diff. equ. => Lie theory

3) <u>normalization</u>
reduce the representation to extremal points of the orbits, e.g. to the point of maximal curve (balance point standardization, FD)

Example for the differential approach



$X(u,v) = X(r,\varphi)$ gray scale image

$$u = r\cos(\varphi)$$

$$v = r\sin(\varphi)$$

Sought are invariants for the rotation group $\mathcal{G}(\varphi)$:

$$g(\varphi)X(u,v) = X(u\cos\varphi - v\sin\varphi, u\sin\varphi + v\cos\varphi)$$

Sought is a function f with property:

$$\frac{df(g(\varphi)X(u,v))}{d\varphi} = 0$$

$$\Rightarrow \frac{\partial f}{\partial u}\frac{\partial u}{\partial \varphi} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial \varphi} = 0 \quad \text{chain rule}$$

$$\Rightarrow \frac{\partial f}{\partial u}(-u\sin\varphi - v\cos\varphi) + \frac{\partial f}{\partial v}(u\cos\varphi - v\sin\varphi) = 0$$

$$\varphi = 0: \quad \boxed{-v\frac{\partial f}{\partial u} + u\frac{\partial f}{\partial v} = 0}$$

This partial differential equation is solved by:

$$f(u,v) = u^2 + v^2 = r^2$$

I.e. all functions, that only depend on radius *r*, and not on angle φ, are feasible invariant features for the rotation group, e.g. the integral over a segment of a circle, or all moments of higher degree:

$$M_n = \int_{r=0}^{R} \int_{\varphi=0}^{2\pi} r^n X(r,\varphi) d\varphi dr$$

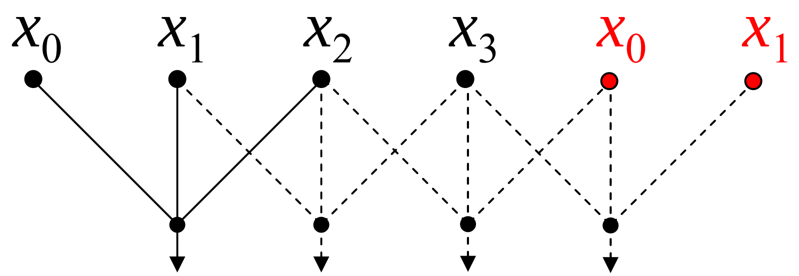
These moments are invariant, but not complete.

The complexity of the differential method lies within the necessity of solving partial differential equations (if the group has more than one parameter).

Integration over the finite group of translations

For finite groups the integration is equal to a summation over the group (socalled group mean). Polynomial functions of finite support) are preferred, e.g. for N=4:

$$\mathbf{x} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$



$$f_1(\mathbf{x}) = x_0 x_1^2 x_2$$

A monom is a polynom

$$P(\mathbf{x}) = x_0^{d_0} x_1^{d_1} \cdots x_{n-1}^{d_{n-1}}$$

The sum $d = \sum_{i=0}^{n-1} d_i$ is called the degree of the monom.

Summation over the group results in a invariant feature:

$$\tilde{x}_1(f_1(\mathbf{x})) = x_0 x_1^2 x_2 + x_1 x_2^2 x_3 + x_2 x_3^2 x_0 + x_3 x_0^2 x_1$$

Verifying the invariance can be done simply by cyclic permutation of the indices: $f(\tau_1(\mathbf{x})) = f(\mathbf{x})$

An even more simple invariant results from summation over the monomial of first degree $f_0 = x_0$:

$$\tilde{x}_0(f_0(\mathbf{x})) = x_0 + x_1 + x_2 + x_3 \quad \text{(simple mean value)}$$

Completeness for finite groups

(Emmy Noether, 1916)

1882-1935

Summing all monomials of degree $\leq|\mathcal{G}|$ results in complete features, or a basis, for finite groups $\mathcal{G}$ with $|\mathcal{G}|$ elements and patterns of dimension N. The number of monomials is given by:

$$\binom{N+|\mathcal{G}|}{N}$$

This is an upper bound, which guarantees completeness; in practice completeness can exist for significantly less elements.

For the translation group results with dim(x)= $|\mathcal{G}|$=N:

$$\binom{2N}{N} = \frac{(2N)!}{(N!)\cdot(N!)}$$

For $N=4$: $\frac{8!}{(4!)\cdot(4!)} = 70$ and for $N=8$ already: $\frac{16!}{(8!)^2} = 12.870$

Orbits for binary patterns of dimension N=4 wrt. cyclic translations

orbits	patterns			
O_0	$(0,0,0,0)^T$			
O_1	$(0,0,0,1)^T$	$(0,0,1,0)^T$	$(0,1,0,0)^T$	$(1,0,0,0)^T$
O_2	$(0,0,1,1)^T$	$(0,1,1,0)^T$	$(1,1,0,0)^T$	$(1,0,0,1)^T$
O_3	$(0,1,0,1)^T$	$(1,0,1,0)^T$		
O_4	$(0,1,1,1)^T$	$(1,1,1,0)^T$	$(1,1,0,1)^T$	$(1,0,1,1)^T$
O_5	$(1,1,1,1)^T$			

Translation invariance of binary patterns of length N=4

Compute all group means of degree ≤ 4 (for reasons of completeness). monomials, that derive from a cyclic transformation of the input values, can be ignored, due to averaging cyclic transformations. Considering also

$$x_i^{d_j} = x_i \quad \text{for } \forall d_j > 0,\; x_i \in \{0,1\}$$

leads to examing the following monomials:

$$f_0(\mathbf{x}) = x_0$$
$$f_1(\mathbf{x}) = x_0 x_1$$
$$f_2(\mathbf{x}) = x_0 x_2$$
$$f_3(\mathbf{x}) = x_0 x_1 x_2$$
$$f_4(\mathbf{x}) = x_0 x_1 x_2 x_3$$

obviously $f_5(\mathbf{x})$=1 is of little practical use

$x_0 x_1 x_2$ e.g. results from cycl. transl. of $x_0 x_2 x_3 = x_{\langle 0+2\rangle} x_{\langle 1+2\rangle} x_{\langle 2+2\rangle}$

Group means

The following group means can be computed from these monomials:

$$\tilde{x}_0 = T(f_0(\mathbf{x})) = (x_0 + x_1 + x_2 + x_3)$$

$$\tilde{x}_1 = T(f_1(\mathbf{x})) = (x_0 x_1 + x_1 x_2 + x_2 x_3 + x_3 x_0)$$

$$\tilde{x}_2 = T(f_2(\mathbf{x})) = (x_0 x_2 + x_1 x_3 + x_2 x_0 + x_3 x_1) = 2(x_0 x_2 + x_1 x_3)$$

$$\tilde{x}_3 = T(f_3(\mathbf{x})) = (x_0 x_1 x_2 + x_1 x_2 x_3 + x_2 x_3 x_0 + x_3 x_0 x_1)$$

$$\tilde{x}_4 = T(f_4(\mathbf{x})) = 4 x_0 x_1 x_2 x_3$$

The following invariants result for the different orbits:

Orbits			$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$
O_0	0	0	0	0	0
O_1	1	0	0	0	0
O_2	2	1	0	0	0
O_3	2	0	2	0	0
O_4	3	2	2	1	0
O_5	4	4	4	4	4

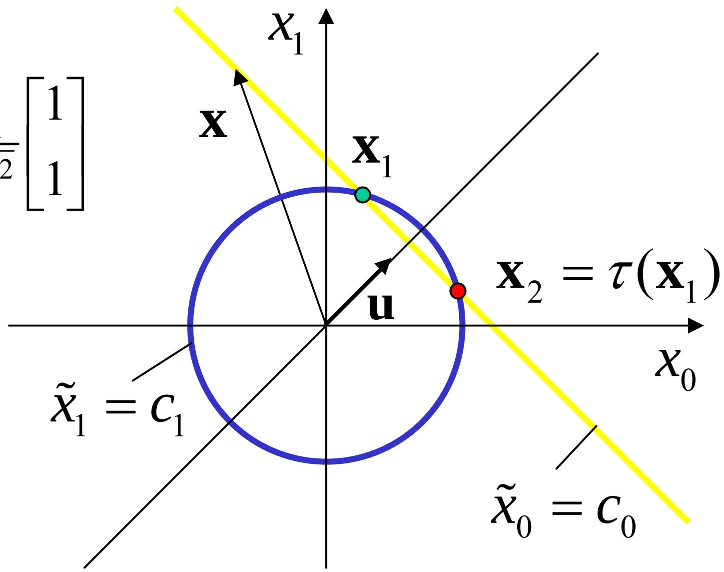
The features $\tilde{x}_0$ and $\tilde{x}_1$ span a complete feature space!

Illustrating the group means geometrically : *intersection of manifolds*

Group mean for the *translation* with N=2:

orbit: $O\left(\underbrace{\begin{bmatrix}x_0\\x_1\end{bmatrix}}_{\mathbf{x}_1}\right)=\left\{\underbrace{\begin{bmatrix}x_0\\x_1\end{bmatrix}}_{\mathbf{x}_1},\underbrace{\begin{bmatrix}x_1\\x_0\end{bmatrix}}_{\mathbf{x}_2}\right\}$ reflecting on bisecting line of 1st quadrant

$\mathbf{u}=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$



1. geometric position with linear group means $f_0=x_0$:

$\tilde{x}_0=x_0+x_1=\sqrt{2}\langle\mathbf{x},\mathbf{u}\rangle=c_0$ (straight line ⊥ to **u**)

still ambigous!

2. geometric position with square group means $f_1=x_0^2$:

$\tilde{x}_1=x_0^2+x_1^2=c_1$ (circle)

intersection of both geometric locations results in exactly the vectors of the equivalence class:

$\Rightarrow\ \tilde{\mathbf{x}}=\begin{bmatrix}\tilde{x}_0\\\tilde{x}_1\end{bmatrix}$ is complete!

Another *linear* invariant does not solve the problem!

Choosing e.g. another multi linear form like $f_1 = 2x_0 + x_1$ results in (after forming the group mean): $\tilde{x}_1 = (2x_0 + x_1) + (2x_1 + x_0) = 3(x_0 + x_1)$

This invariant does not feature new quality compared to:

$\tilde{x}_0 = x_0 + x_1 =\sim \tilde{x}_1$

Choosing an alternative manifold of degree 2

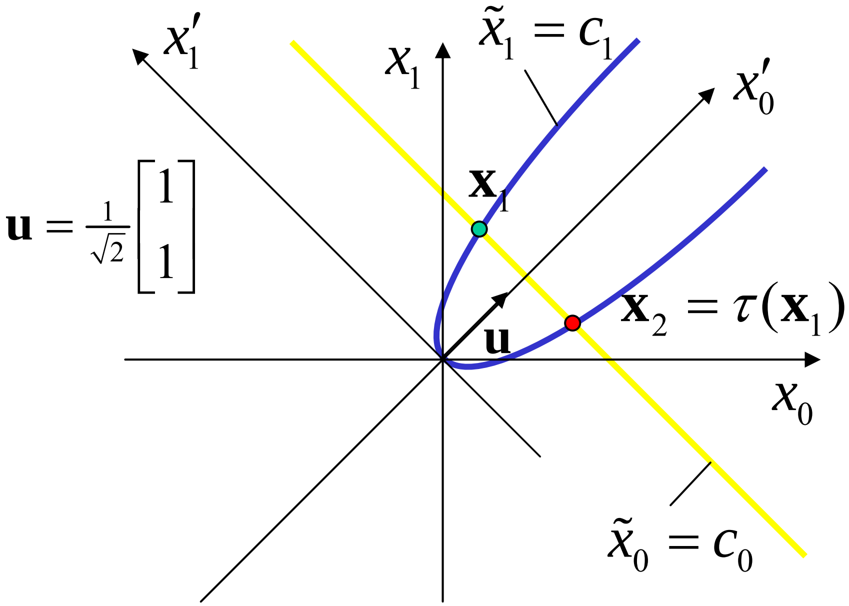
1. geometric location with linear group means $f_0 = x_0$:

$\tilde{x}_0 = x_0 + x_1 = \sqrt{2}\langle \mathbf{x}, \mathbf{u} \rangle = c_0$ (straight line $\perp$ to $\mathbf{u}$) still ambigous!

2. geometric location with parabola symmetric to x_0'-axis: $x_0' = k x_1'^2$

and with $\begin{aligned} x_0' &= 1/\sqrt{2}\,(x_0 + x_1) \\ x_1' &= 1/\sqrt{2}\,(x_1 - x_0) \end{aligned}$ follows: $x_0^2 + x_1^2 - 2x_0x_1 - 1/k\,(x_0 + x_1) = 0$

This can be prodiced as a square group mean over $f_1 = 1/2\,(x_0^2 + x_1^2) - x_0x_1 - \frac{1}{2k}x_0$



$$\begin{array}{c} 1/2\,(x_0^2 + x_1^2) - x_0x_1 - \frac{1}{2k}x_0 \\ 1/2\,(x_0^2 + x_1^2) - x_0x_1 - \frac{1}{2k}x_1 \\ \hline \tilde{x}_1 = x_0^2 + x_1^2 - 2x_0x_1 - 1/k\,(x_0 + x_1) \end{array}$$

intersection of both geometric locations results in exactly the vectors of the equivalence class:

$\Rightarrow \quad \tilde{\mathbf{x}} = \begin{bmatrix}\tilde{x}_0 \\ \tilde{x}_1\end{bmatrix}$ is complete!

Group mean for the equivalence class of cyclic translations with N=3

orbit: $O\left(\underbrace{\begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}_1}\right) = \left\{ \underbrace{\begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}_1}, \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_0 \end{bmatrix}}_{\mathbf{x}_2}, \underbrace{\begin{bmatrix} x_2 \\ x_0 \\ x_1 \end{bmatrix}}_{\mathbf{x}_3} \right\}$ rotation at 1st space diagonal about $2\pi/3$

1. geometric location with linear group means $f_0 = x_0$:

$\tilde{x}_0 = x_0 + x_1 + x_2 = \sqrt{3}\langle \mathbf{x}, \mathbf{u} \rangle = c_0$ (plane $\perp$ to $\mathbf{u} = \frac{1}{\sqrt{3}}[1\ 1\ 1]^T$)

2. geometric location with square group means $f_1 = x_0^2$:

$\tilde{x}_1 = x_0^2 + x_1^2 + x_2^2 = c_1$ (sphere) $\tilde{x}_0 \cap \tilde{x}_1$ results in circle

3. geometric location:

a) $f_2 = x_0 x_1 \Rightarrow$ group mean: $\tilde{x}_2 = x_0 x_1 + x_1 x_2 + x_2 x_0 = c_2$

$\tilde{x}_0 \cap \tilde{x}_1 \cap \tilde{x}_2$ results in 6 points (also invariant to reflection!)

b) completeness not until choosing an asymmetric function:

$f_2 = x_0^2 x_1 \Rightarrow$ group mean: $\tilde{x}_2 = x_0^2 x_1 + x_1^2 x_2 + x_2^2 x_0 = c_2$

$\tilde{x}_0 \cap \tilde{x}_1 \cap \tilde{x}_2$ results in exactly the 3 points of the equivalence class!

start Matlab demo
matlab-MF.bat

Group mean for the equivalence class of cyclic translations with N=3

$$\text{orbit: } O\left(\underbrace{\begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}_1}\right) = \left\{ \underbrace{\begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}_1}, \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_0 \end{bmatrix}}_{\mathbf{x}_2}, \underbrace{\begin{bmatrix} x_2 \\ x_0 \\ x_1 \end{bmatrix}}_{\mathbf{x}_3} \right\} \text{ rotation at 1st space diagonal about } 2\pi/3$$

1. geometric location with linear group mean $f_0 = x_0$:

$\tilde{x}_0 = x_0 + x_1 + x_2 = \sqrt{3}\langle \mathbf{x}, \mathbf{u} \rangle = c_0$ (plane $\perp$ to $\mathbf{u} = \frac{1}{\sqrt{3}}[1\ 1\ 1]^T$)

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$\tilde{x}_0 \cap \tilde{x}_1 \cap \tilde{x}_2$ results in exactly the 3 points of the equivalence class!

start Matlab demo
matlab-MF.bat

Intersection of manifolds

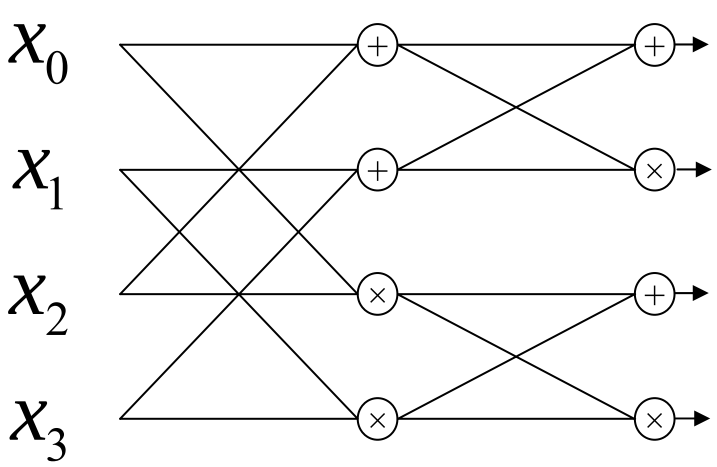
$$\begin{array}{ccc} \tilde{x}_0(x_0,x_1,x_2)=c_0 & \Rightarrow \text{ 1st hyper area} & g_0(x_0,x_1,x_2,c_0)=0 \\ \tilde{x}_1(x_0,x_1,x_2)=c_1 & \Rightarrow \text{ 2nd hyper area} & g_1(x_0,x_1,x_2,c_1)=0 \\ \vdots & \vdots & \vdots \end{array}$$

$$g_0 \cap g_1 \cap g_2 \cdots$$

- The necessary constraint guarantees that all manifolds go through the points of the equivalence class.
- Adding more and more independent invariants, normally the intersection is reduced, which leads to a higher degree of completeness (the hyper areas subtend in less points and resp. in manifolds of lower degree).

Relations to class ℂT

If invariants of class ℂT are calculated with functions $f_1=a+b$ and $f_2=a\times b$, for N=4 results:



$$\tilde{x}_0 = (x_0 + x_1 + x_2 + x_3)$$
$$\tilde{x}_1 = (x_0 x_1 + x_1 x_2 + x_2 x_3 + x_3 x_0)$$
$$\tilde{x}_2 = (x_0 x_2 + x_1 x_3)$$
$$\tilde{x}_3 = x_0 x_1 x_2 x_3$$

This is exactly a subset of invariants, that can be derived via forming group means over monomials, but is computed with a fast algorithm of complexity $O(N \log N)$ instead of $O(N^2)$.

Constructing weak commutative preprocessing mappings

Lemma:

Every linear combination of weak commutative mappings is weak commutative.

Also compare modus operandi defining $\mathbb{C}T_2$ and $\mathbb{C}T_3$ with generally *compatible permutations* ω_i as feasible preprocessing:

$$\mathbb{C}T_3 := \underbrace{\mathbb{C}T_{zs} \cup \mathbb{C}T_{sz}}_{\mathbb{C}T_2} \cup \mathbb{C}T_{DI}$$

$$\Rightarrow \mathbb{I}_{\mathbb{C}T_3} := \mathbb{I}_{\mathbb{C}T_{zs}} \cap \mathbb{I}_{\mathbb{C}T_{sz}} \cap \mathbb{I}_{\mathbb{C}T_{DI}}$$

with:

$$\mathbb{C}T_{sz} = \mathbb{C}T_{zs} \circ \omega_T \circ \mathbf{X}$$

$$\mathbb{C}T_{DI} = \mathbb{C}T_{zs} \circ \omega_{DI} \circ \mathbf{X}$$

Invariants for the group of cyclic translations for binary patterns of dimension N=16

$2^{16} = 65.536$ different binary patterns result.

Using the Pólya-theory it can be shown, that there exists exactly the following number of distinguishable orbits or equivalence classes:

$$A_B = \frac{1}{N} \sum_{k|N} \varphi(k) 2^{\frac{N}{k}} \Bigg|_{N=16} = 4116$$

The sum is over all factors k of N. φ is the Euler totient function φ (The Euler totient φ(n) computes the number of all positive integers k with $1 \leq k \leq n$ and coprime to n).

Discrimination properties of polynomial features

The following 9 monomials have been averaged over the group:

$$f_0 = x_0 \qquad f_3 = x_0 x_3 \qquad f_6 = x_0 x_6$$
$$f_1 = x_0 x_1 \qquad f_4 = x_0 x_4 \qquad f_7 = x_0 x_7$$
$$f_2 = x_0 x_2 \qquad f_5 = x_0 x_5 \qquad f_8 = x_0 x_8$$

The following dicrimination properties result:

With A_s equal to number of separable equivalence classes, the equivalent of number of feature vectors different from each other.

feature set	sep. patterns A_S	$\Delta = A_S / A_B$
$\tilde{x}_0$	17	0,004
$\{\tilde{x}_0, \tilde{x}_1\}$	66	0,016
$\{\tilde{x}_0, \tilde{x}_1, \tilde{x}_2\}$	200	0,049
$\{\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3\}$	501	0,122
$\{\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4\}$	980	0,238
$\{\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5\}$	1516	0,368
$\{\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5, \tilde{x}_6\}$	1818	0,442
$\{\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5, \tilde{x}_6, \tilde{x}_7\}$	1876	0,456
$\{\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5, \tilde{x}_6, \tilde{x}_7, \tilde{x}_8\}$	1876	0,456

Improving the discrimination properties of polynomial features

With these two features:

$$\tilde{x}_0 = x_0 + \ldots + x_{15}$$

$$\tilde{x}_1 = (x_0 + x_1)^2 + (x_1 + x_2)^2 + \ldots + (x_{15} + x_0)^2$$

and the weak commutative mapping:

$$(\omega \mathbf{x})_i = x_i + (2x_{(i-1) \bmod 16} + x_{(i+1) \bmod 16})^2, \quad 0 \le i \le 15$$

result these new discrimination properties:

After already 4 iterations of the mapping ω the corresponding feature set is complete!

feature set	sep. patterns	$\Delta = A_s / A_B$
$\{\tilde{x}_0, \tilde{x}_1\}$	66	0,02
$\{\tilde{x}_0 \circ \omega^1, \tilde{x}_1 \circ \omega^1\}$	906	0,22
$\{\tilde{x}_0 \circ \omega^2, \tilde{x}_1 \circ \omega^2\}$	3630	0,88
$\{\tilde{x}_0 \circ \omega^3, \tilde{x}_1 \circ \omega^3\}$	4086	0,99
$\{\tilde{x}_0 \circ \omega^4, \tilde{x}_1 \circ \omega^4\}$	4116	1,00

Improving the discrimination properties for transformations of class $\mathbb{C}T$

for binary patterns

transformations	RT	(+,x)	BT	$\|F\|$
separable patterns A_s	225	230	168	1876
$\Delta = A_s / A_B$	0,055	0,056	0,041	0,456

transformations	separable patterns	Δ
RT	225	0,05
$RT \circ \omega_1$	3682	0,89
$RT \circ \omega_1 \circ \omega_1$	4116	1,00
$RT \circ \omega_2$	4088	0,99
$RT \circ \omega_2 \circ \omega_2$	4116	1,00
$RT \circ \omega_3$	4116	1,00

With the following weak commutative mappings:

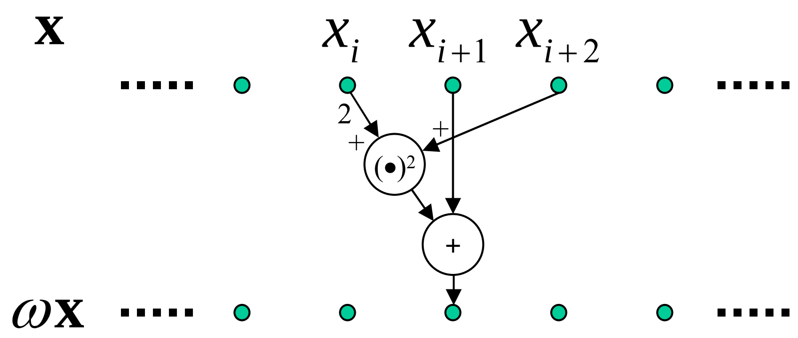
$$(\omega_1 \mathbf{x})_i = x_i + \left(\sum_{i=0}^{15} x_i \right) (2 x_{(i+1) \bmod 16} + x_{(i+2) \bmod 16})^2$$

$$(\omega_2 \mathbf{x})_i = x_i + (x_{(i+1) \bmod 16} + 2 x_{(i+2) \bmod 16} + 3 x_{(i+3) \bmod 16})^2$$

$$(\omega_3 \mathbf{x})_i = x_i + (x_{(i+1) \bmod 16} + 2 x_{(i+2) \bmod 16} + 3 x_{(i+3) \bmod 16} + 4 x_{(i+4) \bmod 16})^2$$

These maps are chosen mostly arbitrarily!

for example:



In addition can be shown that for the group of cyclic transformations of length N with maximal $(N+1)$ properly chosen features completeness can be obtained.

Bu5

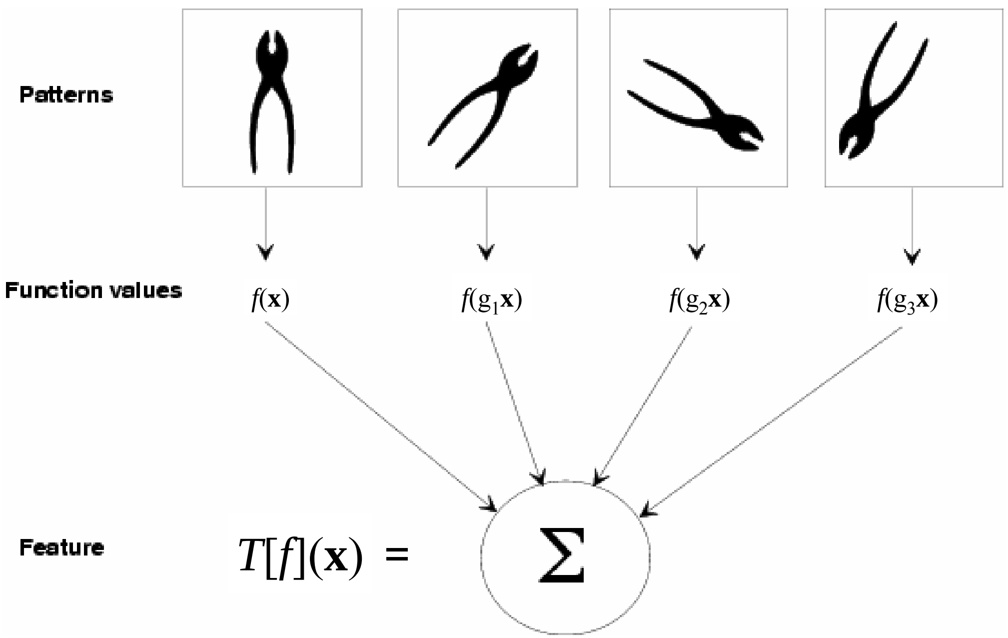
Folie 25

Bu5 Beweis dazu in DFG-Report von Herrn Canterakis
Burkhardt; 19.11.2002

Invariants derived from computing the group mean over *Euclidian motion*

Invariants can be obtained using arbitrary functions f over integration over the motion group:

$$I[f](\mathbf{x}) = \int_G f(g\mathbf{x})dg$$



Integral invariants for the group of plane motions with *local* functions

For the cyclic plane motion applies:

$$g(t_0, t_1, \varphi)\mathbf{x}[i, j] = \mathbf{x}[k, l]$$

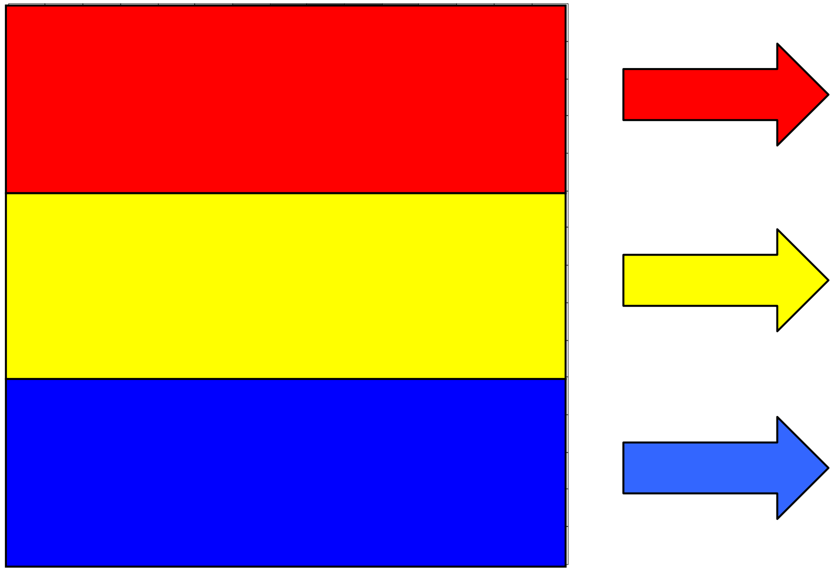
with:

$$\begin{pmatrix} k \\ l \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} i \\ j \end{pmatrix} - \begin{pmatrix} t_0 \\ t_1 \end{pmatrix}$$

all indices to be understood modulo the image dimensions.

$$T[f](\mathbf{x}) = \frac{1}{2\pi NM} \int_{t_0=0}^{N} \int_{t_1=0}^{M} \int_{\varphi=0}^{2\pi} f(g\mathbf{x})\, d\varphi\, dt_1 dt_0$$

Which proper kernel functions to choose? Demo: linear feature kernel functions for a rotating handwritten numerals



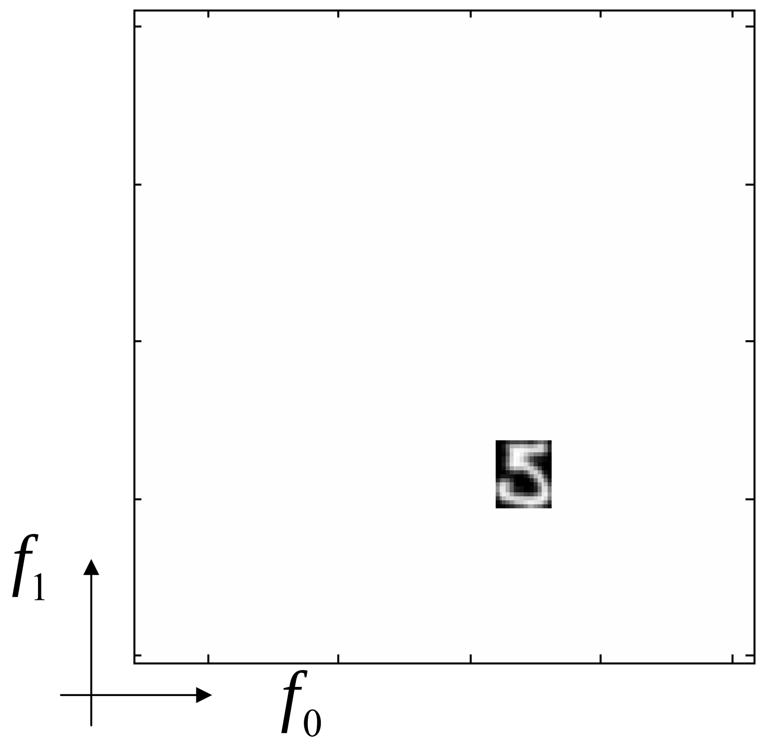
$$f_0 = \sum_{0 \le i \le N/3-1} \; \sum_{0 \le j \le N-1} x_{i,j}$$

$$f_1 = \sum_{N/3 \le i \le 2N/3-1} \; \sum_{0 \le j \le N-1} x_{i,j}$$

$$f_2 = \sum_{2N/3 \le i \le N-1} \; \sum_{0 \le j \le N-1} x_{i,j}$$

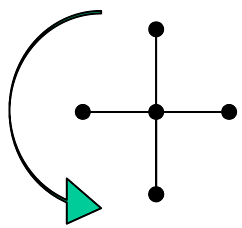
R. Herbrich (2004)

Orbits of features



orbit\orbit.html

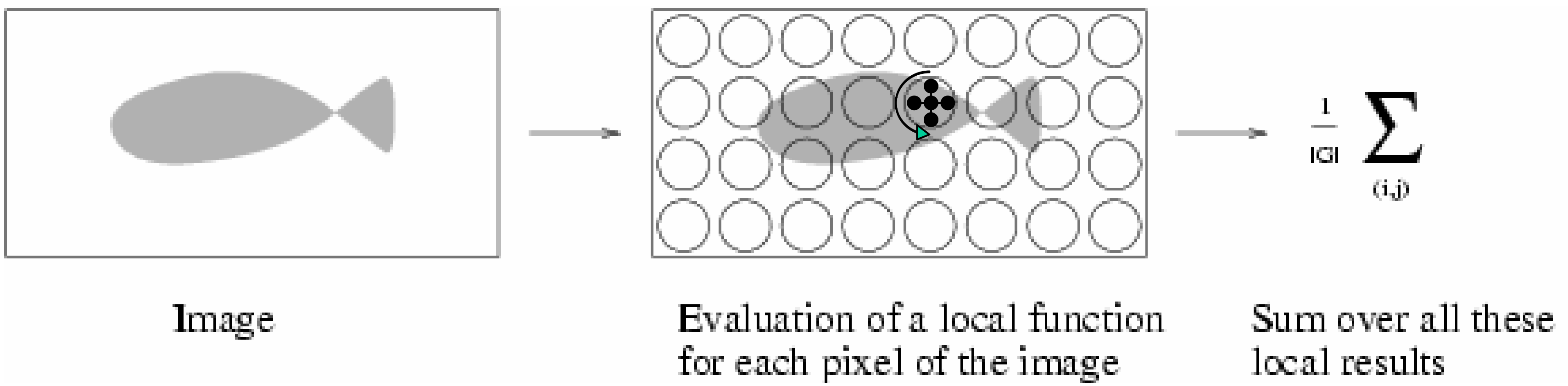
f and *g* are ***interconvertible***, i.e. the generally local function is evaluated for the whole image with an Euclidian motion. The integration is approximated by a summation on the pixel raster and a rotation about a finite number of angles with bilinear interpolation of interim values.

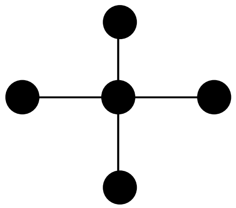


Calculating the monomial:

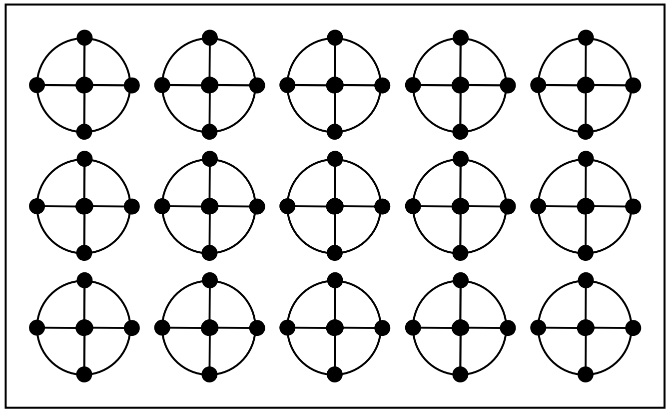
	$x_{1,0}^{3}$	
$x_{0,-1}^{1}$	$x_{0,0}^{5}$	$x_{0,1}^{2}$
	$x_{-1,0}^{2}$	

$$= x_{0,-1}^{1} \cdot x_{1,0}^{3} \cdot x_{0,0}^{5} \cdot x_{-1,0}^{2} \cdot x_{0,1}^{2}$$

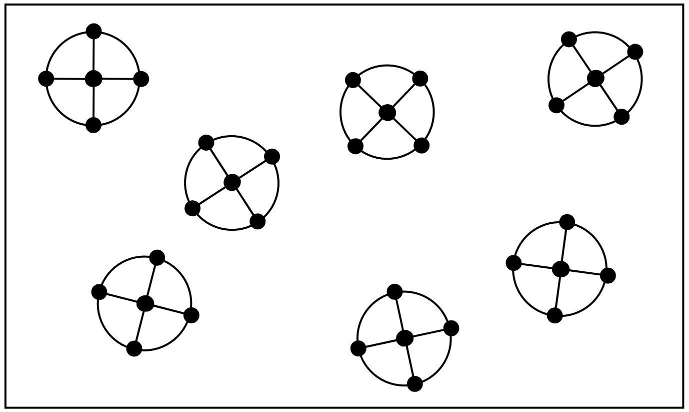




monomial

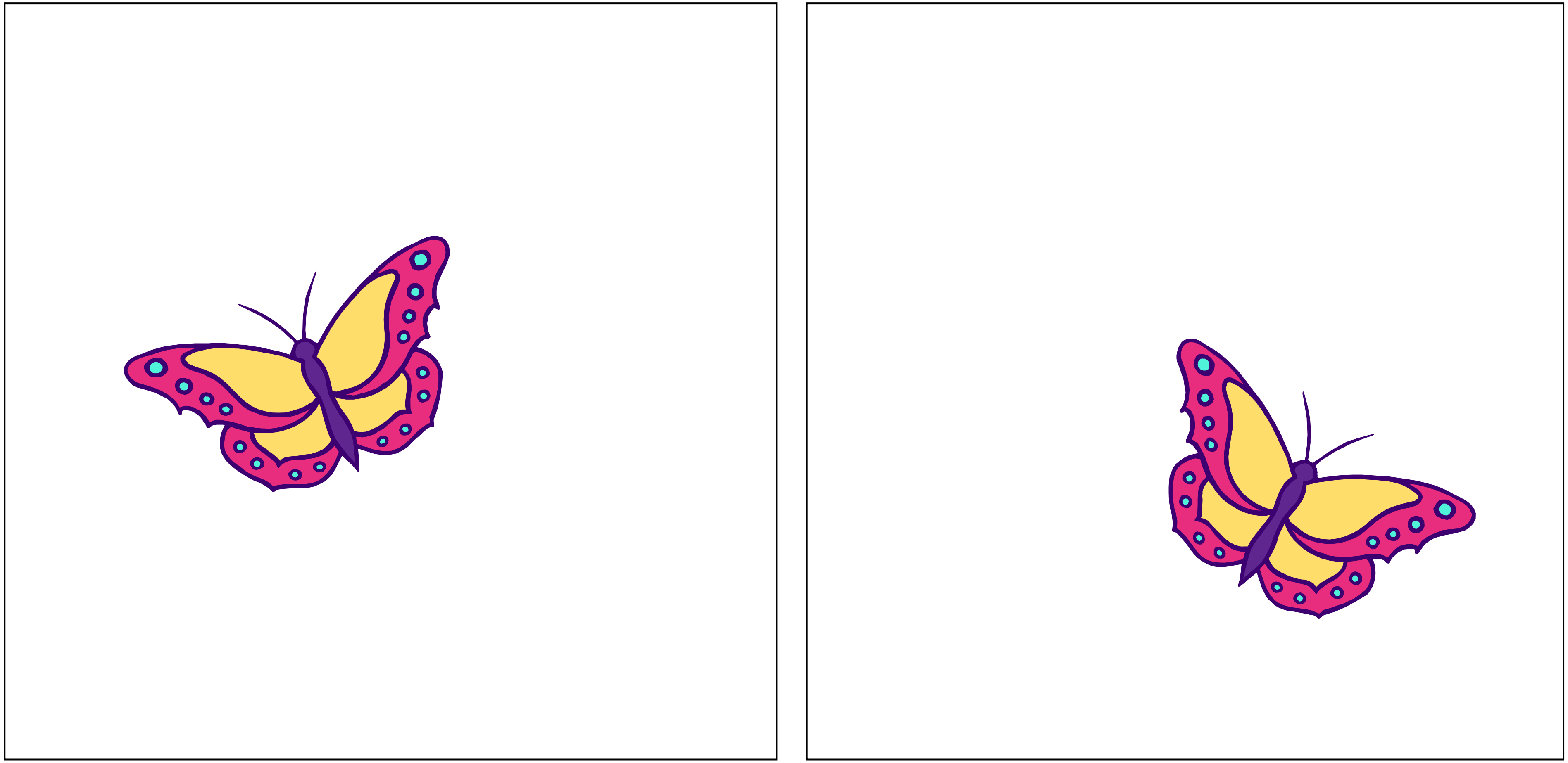


Deterministic integral over Euclidian planar motion

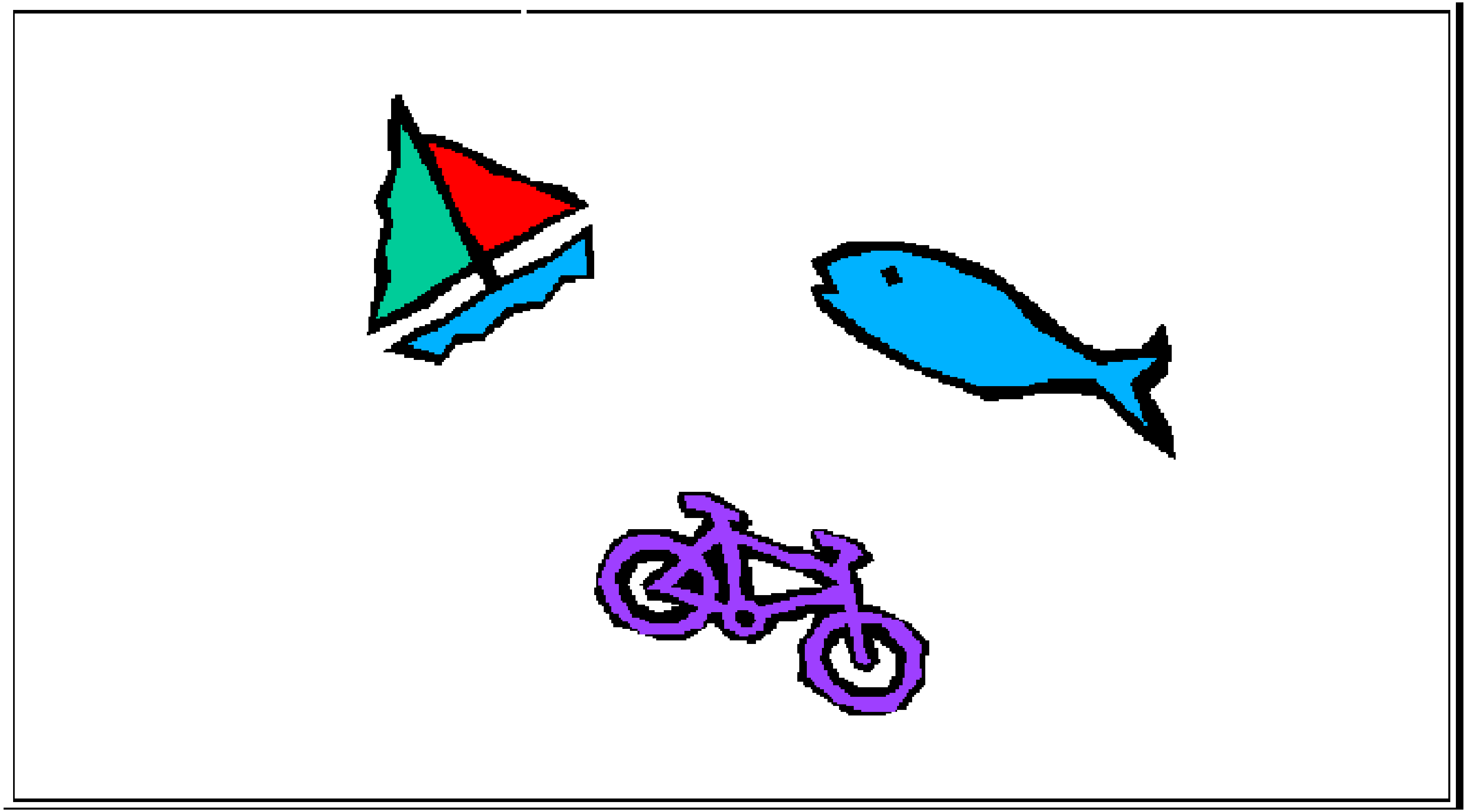


Monte Carlo integration

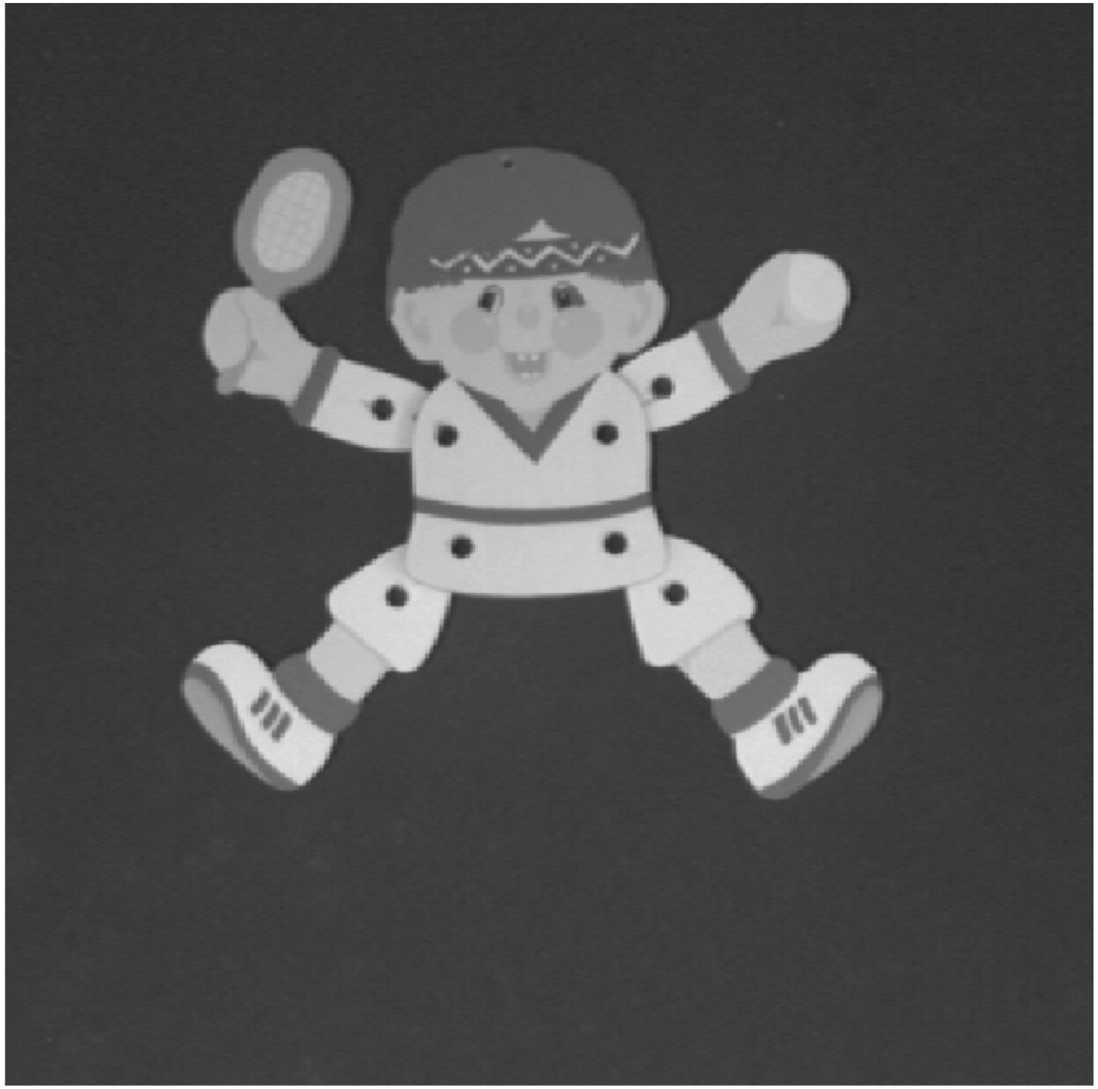
Invariance wrt. a global Euclidian motion (translation and rotation)



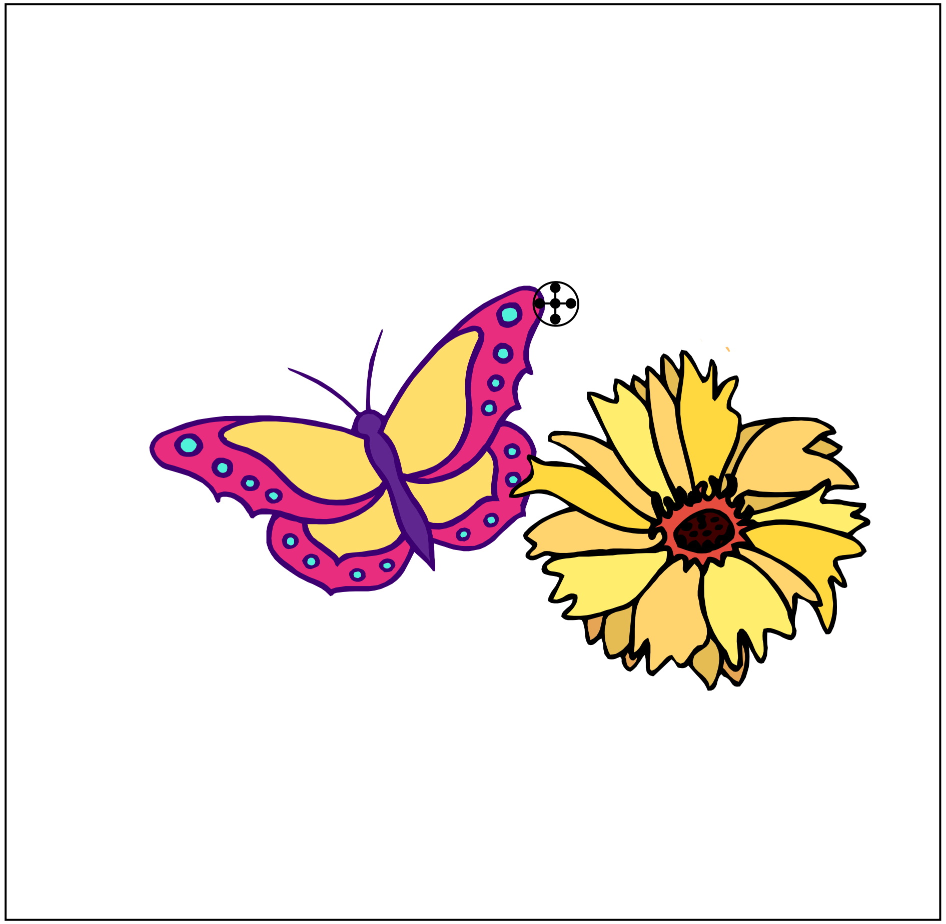
Equivalence class for objects with independent Euclidian motion (using functions with local domain)

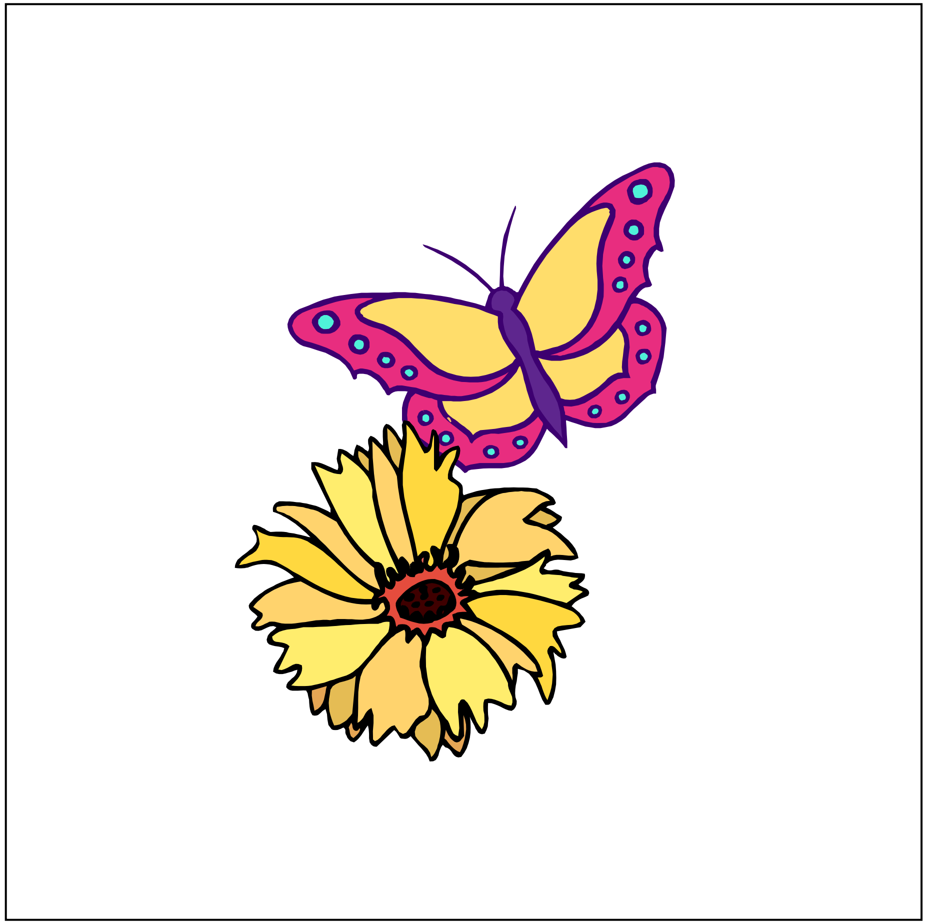


Recognition of objects with joints



Recognition of two objects in a scene without segmenting





Robustness against topological deformations