**Algorithms for Picture Analysis**
**Prof. Reinhard Klette**
**SS 05**

**Coursework 5**

## 5.2.(i) [Outer Polygons-implicit formula ]

In order to construct the outer $n_m$-gon $Q_{n_m}$ circumscribing a circle, we must consider the perimeter $P(Q_{n_m}) = n_m \cdot f_m$ of this polygon. $n_m = 3 \cdot 2^m$ describes the number of edges of length $f_m$.

For $m = 0$ we get $n_m = 3$ and therfore the outer equilateral triangle as in Figure 1. For $m = 1$ we get $n_m = 6$ and therfore the outer hexagon as in Figure 2.

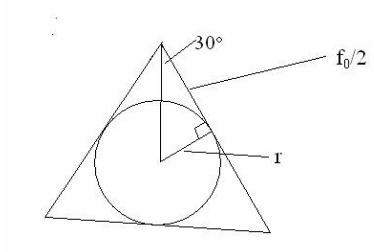


Fig1

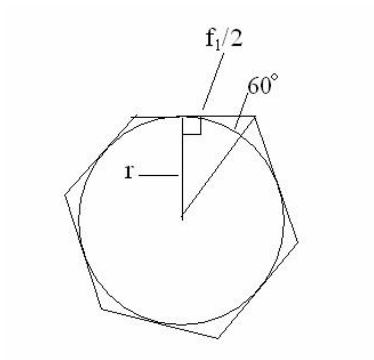


Fig2

By Figure 1 (m=0,$\alpha = 30°$), we have:

$$\tan 30° = \frac{r}{\frac{f_0}{2}} \Rightarrow f_0 = \frac{2r}{\tan 30°} = 2 \cdot r \cdot \sqrt{3}.$$

By Figure 2 ($\alpha = 60°$), we have:

$$\tan 60° = \frac{r}{\frac{f_1}{2}} \Rightarrow f_1 = \frac{2r}{\tan 60°} = \frac{2r}{\sqrt{3}}.$$

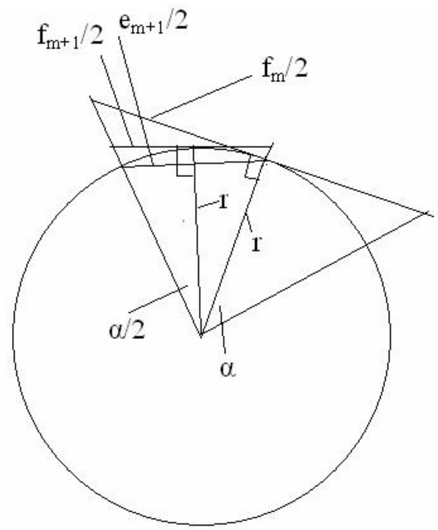


Fig3

From this beginning cases we can conclude that by incrementing m by 1 the angle $\alpha$ is cut in half. So for the general case (see Figure 3), we have:

$$\tan \alpha = \frac{\frac{f_m}{2}}{r} \tag{1}$$

$$\tan \frac{\alpha}{2} = \frac{\frac{f_{m+1}}{2}}{r} \tag{2}$$

$$\sin \frac{\alpha}{2} = \frac{\frac{e_{m+1}}{2}}{r} \tag{3}$$

Note that (addition theorem for tangens):

$$\tan \alpha = \frac{2 \cdot \tan \frac{\alpha}{2}}{1 - \tan \frac{\alpha}{2} \cdot \tan \frac{\alpha}{2}} \tag{4}$$

(Proof:

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)}$$

divide by $\cos\alpha\cos\beta$.)

By (1), (2) and (4), we have:

$$\frac{f_m}{2r} = \frac{\frac{f_{m+1}}{r}}{1-\frac{f_{m+1}}{2\cdot r}\cdot\frac{f_{m+1}}{2\cdot r}}.$$

$$f_{m+1} = \frac{2r(\sqrt{(4r^2+f_m^2)}-2r)}{f_m}$$

**5.2.(ii) [Outer Polygons-explicit formula ]**

Note that

$$\tan\frac{\alpha}{2} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \frac{\sin\frac{\alpha}{2}}{\sqrt{1-\sin\frac{\alpha}{2}\cdot\sin\frac{\alpha}{2}}}$$

By (2) and (3) (part (i)), we have:

$$\frac{f_{m+1}}{2} = \frac{\frac{e_{m+1}}{2r}}{\sqrt{1-\frac{e_{m+1}}{2r}\cdot\frac{e_{m+1}}{2r}}}$$

Therfore,

$$f_{m+1} = \frac{2re_{m+1}}{\sqrt{4r^2-e_{m+1}^2}}$$

where

$$e_{m+1} = r\sqrt{2-\sqrt{2+\sqrt{2+...+\sqrt{2+\sqrt{3}}}}}$$

(See page 13 Lecture 05)

**5.2.(iii) [Approximative calculation of $\pi$ ]**

$$P(Q_{n_0}) = 3 \cdot 2^0 \cdot f_0 = 3 \cdot r \cdot 3.46 = 2 \cdot r \cdot 5.19.$$

$$P(Q_{n_1}) = 3 \cdot 2^1 \cdot f_1 = 6 \cdot r \cdot 1.15 = 2 \cdot r \cdot 3.46.$$

$$P(Q_{n_2}) = 3 \cdot 2^2 \cdot f_2 = 12 \cdot r \cdot 0.54 = 2 \cdot r \cdot 3.22.$$

The perimeter $P(Q_{n_m}) = n_m \cdot f_m$ will approach $2\pi r$ as m continues to increase towards infinity. Note that $n_m = 3 \cdot 2^m$. Let r=1. We have:

$$3 \cdot 2^m \cdot f_m \to 2\pi (m \to \infty).$$

or

$$3 \cdot 2^{m-1} \cdot f_m \to \pi (m \to \infty).$$