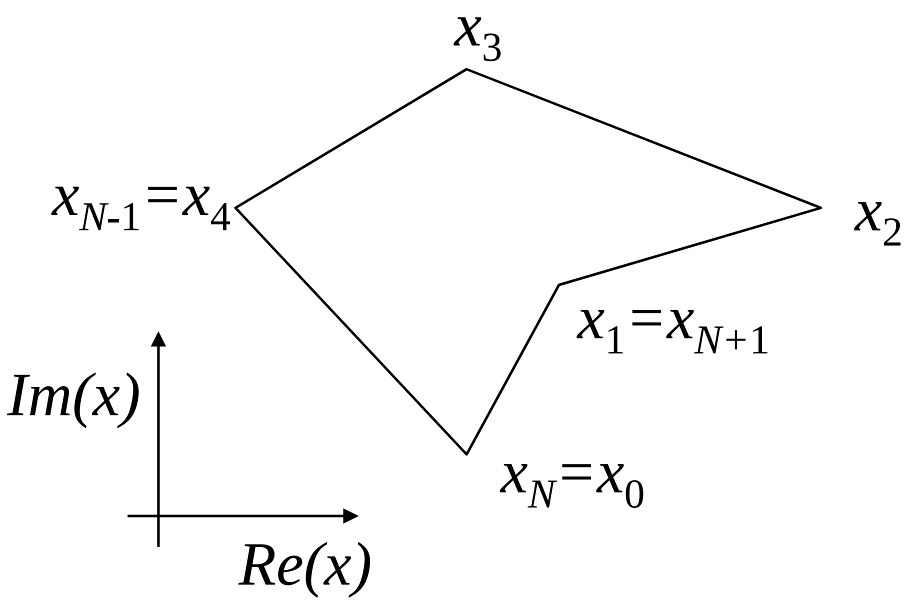
Calculating Fourier coefficients of polygons

The integral of polygons along the straight line can be solved explicitly and formulas with explicit dependency on vertices of the polygon are obtained. Sampling contours usually results in non-equidistant polygon edges. Thus either the Fast Fourier Transform (FFT, see DBV-I) for calculation of the FC cannot be applied directly or additional interpolation of data is needed. The following explicit expression is to be preferred, especially when eliminating the points along the stright lines.

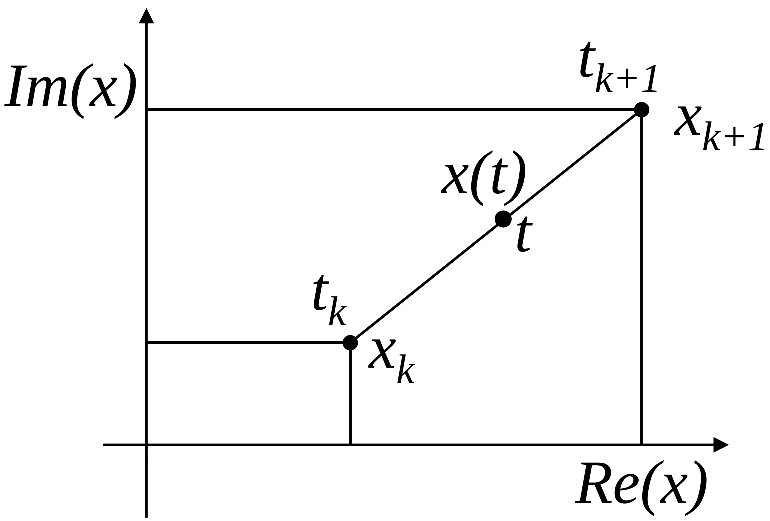


Due to the section-wise linear traversal the integral can be decomposed in:

$$c_n = \frac{1}{T}\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}} x(t)\cdot e^{-jn\omega t}dt$$

with: $e^{-jn\omega t} = \cos(n\omega t) - j\sin(n\omega t)$

Interpolation along the polygon sections results in:



$$x(t) = x_k + (x_{k+1} - x_k)\frac{\Delta t}{t_{k+1} - t_k}$$

with: $\Delta t = t - t_k$

resp.: $\Delta x(t) = x(t) - x_k = \Delta x_k \dfrac{\Delta t}{t_{k+1} - t_k}$

Analytic integration in sections:

$$\int e^{-jn\omega t} dt = \frac{j}{n\omega} e^{-jn\omega t}$$

and:

$$\int t e^{-jn\omega t} dt = \frac{1}{(n\omega)^2} e^{-jn\omega t} + \frac{j}{n\omega} t e^{-jn\omega t}$$

$$= \frac{1}{(n\omega)^2} e^{-jn\omega t} (1 + jn\omega t)$$

For vertices of a closed polygon:

$$x_i = x_k \quad \text{for:} \quad i \equiv k \bmod N$$

the first and second difference elements are computed:

$$\Delta x_i = x_{i+1} - x_i$$

$$\Delta^2 x_i = \Delta x_{i+1} - \Delta x_i = x_{i+2} - 2x_{i+1} + x_i$$

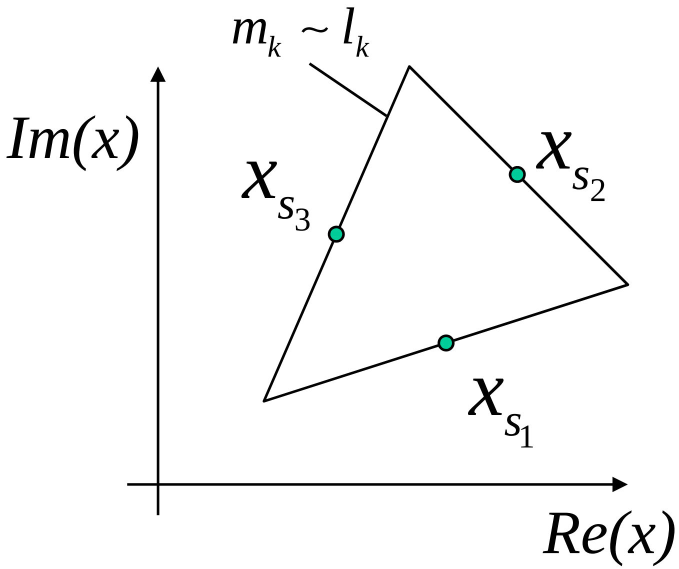
and hence normed differences:

$$\Delta z_i = \Delta x_i / \left|\Delta x_i\right|$$

$$\Delta^2 z_i = \Delta z_{i+1} - \Delta z_i$$

The 0th FC denotes the location of the balance point and can be calculated as follows:

$$c_0 = \frac{1}{2T}\sum_{k=0}^{N-1}(x_k + x_{k+1})\left|\Delta x_k\right|$$



$$x_s = \frac{\sum x_{s_k} \cdot m_k}{\underbrace{\sum m_k}_{\sim \sum l_k = T}} = \frac{1}{T}\sum_{k=0}^{N-1} \underbrace{\frac{(x_k + x_{k+1})}{2}}_{x_{s_k}} \underbrace{\left|x_{k+1} - x_k\right|}_{l_k}$$

$$= \frac{1}{2T}\sum_{k=0}^{N-1}(x_k + x_{k+1})\left|\Delta x_k\right| = c_0$$

All other Fourier coefficients can be calculated for non-equidistant nodes exactly as:

$$c_n = \frac{T}{(2\pi n)^2} \sum_{k=0}^{N-1} (\Delta z_{k-1} - \Delta z_k) e^{-jn(\frac{2\pi}{T})t_k}$$

$$= -\frac{T}{(2\pi n)^2} \sum_{k=0}^{N-1} \Delta^2 z_{k-1} e^{-jn(\frac{2\pi}{T})t_k}$$

With partial arc lengths:

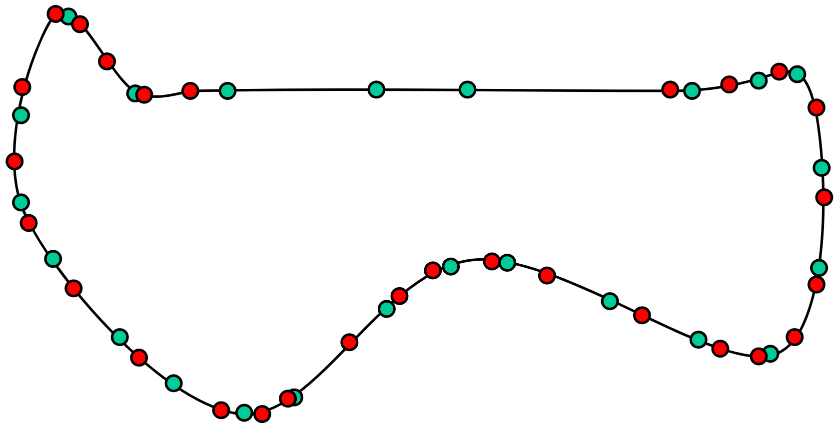
$$t_k = \sum_{i=0}^{k-1} \left| \Delta x_i \right| \qquad k > 0,\ t_0 = 0$$

These characteristic contour values that are derived from second differences closely relate to the fundamental contour description using curve and its calculation using differential quotients.

Representing a contour by a variable set of points

The fact that the number and location of the sampling points along the contour is irrelevant for a fundamental Fourier description matters significantly for practical issues, as long as these sampling points represent the contour sufficiently exactly. In both cases approximately the same FCs result.

Sometimes it would make sense to remove points along straight lines!



The equivalence class of all similar patterns

For the group of similarities the following feasible modifications of a pattern *x(t)* compared to a referenve pattern $x^0(t)$ result:

1. a translation about the complex value z,
2. a rotation about the angle Φ about a fixed reference point x_B,
3. a translation of the starting point about t_0 and
4. a radial dilation about R with regard to x_B.

Applied to the reference pattern:

$$x(t) = x^0(t + t_0) \cdot R \cdot e^{j\Phi} + z \qquad (4.1.5)$$

The equivalence class of all similar Fourier coefficients

The effects of the similarity transform on the Fourier coefficients result due to the linearity of the Fourier transform as follows. The coefficients are calculated assuming a normed arc length $T = 2\pi$.

$$c_0 = c_0^0 \cdot R \cdot e^{j\Phi} + z \qquad (4.1.6)$$

$$c_n = c_n^0 \cdot R \cdot e^{j(\Phi + n \cdot t_0)} \quad \text{for } n \neq 0 \qquad (4.1.7)$$

The Fourier coefficient c_0 only denotes the location of the line balance point and is invariant towards a translation of the starting point. The other Fourier coefficients‘ arguments change depending on rotation and starting point translation and its amplitude depending on an additional dilation. These coefficients are independent from a translation.

Invariance porperties of the power spectrum

Following from eq. 4.1.7 , the amplitude spectrum of the Fourier transformed (the amplitudes of the Fourier coefficients) forms location invariant features regarding the equivalence class of congruent patterns (Euclidian motion). However, only the neccessary condition of invariance is satisfied; completeness is not given, because arbitrary phase modulations are suppressed

See pic. with phase modulation at c_1 (chapter 4c, p. 3).

Similarity invariant Fourier descriptors

As opposed to class $\mathbb{C}$T, a *complete* set of location and scale invariant features (Fourier desrciptors) can be derived from the Fourier coefficients by normalizing its amplitudes and fixing its arguments by a unique side condition.

The FC must yield exactly four degrees of freedom according to the changes (R,Φ,t_0,z). The complex invariants are defined in the following sentence:

Sentence: Regarding the complex pattern class $\mathrm{x} \in \mathbb{F}^N$ (band limited contour patterns) form the complex features

$$\{\tilde{x}_n := \frac{|c_n|}{|c_q|} e^{j(\Phi_n + \alpha\Phi_r - \beta\Phi_q)}\} \quad n = \pm 1, \pm 2, \cdots, \pm N/2 \qquad (4.2.1a)$$

$$\alpha = \frac{q - n}{r - q} \qquad \beta = \frac{r - n}{r - q}$$

$$r = q + s \qquad q \in \mathbb{N} \qquad (4.2.1b)$$

$s \triangleq$ degree of rotation symmetry

(solely $\tilde{x}_q$ and $\mathrm{Im}(\tilde{x}_r)$) a *complete*, *minimal* set of invariants with respect to the group of plane motions (translation and rotation) and the radial dilation (group of similarities), with a reference pattern normed to one of the following values

$$x_B^0 = 0 \quad (c_0^0 = x_s) \qquad (4.2.1c)$$

(i.e. the fixed reference point lies on the origin)

$$|c_q^0| = 1 \qquad (4.2.1d)$$

$$\Phi_q^0 = 0 \qquad (4.2.1d)$$

$$\Phi_r^0 = 0 \qquad (4.2.1d)$$

The set of invariants is minimal, i.e. discarding one feature, completeness with regard to $\mathbb{F}^N$ is lost.

Reducing the degrees of freedom $\{\Phi_q, \Phi_r, |c_q|, \mathrm{Re}(c_0), \mathrm{Im}(c_0)\}$ exactly about the amount of the degrees of freedom of the pattern space $\{t_0, R, \Phi, \mathrm{Re}(z), \mathrm{Im}(z)\}$ shows the minimality of the invariants.

Proof:

Proof by construction, by reconstructing all characteristic values of the reference pattern from the invariants in a unique way.

The complexity of the phase normalization actually results from the ambiguity of the phase mod(2π); which means that it cannot be reproduced how many times 2π has been passed through.

The normalization conditions for the reference pattern as stated in (4.2.1c-f) determine exactly the four degrees of freedom location (z), quantity (R), orientation (Φ) and model/space point translation (t_0).

For the amplitude of the invariance results from (4.1.7) and (4.2.1d)

$$|\tilde{x}_n| = \frac{|c_n|}{|c_q|} = \frac{|c_n^0| \cdot \cancel{R}}{|c_q^0| \cdot \cancel{R}} = |c_n^0| \qquad (4.2.2)$$

A unique relation between: $|\tilde{x}_n|$ and $|c_n^0|$

In some practical cases scale invariance is undesired (e.g. quality control). In these cases the amplitudes can be defined by:

$$\left|\tilde{x}_n\right| = \left|c_n\right| \qquad (4.2.3)$$

A general ***linear form*** for elimination is to be used, because the arguments of the Fourier coefficients according to (4.1.7) are affected additively by feasible changes. Considering that all arguments of the Fourier coefficients of an unknown pattern *x(t)* are only known mod(2π), and considering for amplitudes an accordingly multiple g_i of 2π , results from (4.2.1a) and taking into account eq. (4.1.7)

$$\begin{aligned}\arg(\tilde{x}_n) &= \Phi_n + \alpha\Phi_r - \beta\Phi_q \\ &= \Phi_n^0 + \Phi + n \cdot t_0 + g_n \cdot 2\pi \\ &\quad +\alpha(\Phi_r^0 + \Phi + r \cdot t_0 + g_r \cdot 2\pi) \\ &\quad -\beta(\Phi_q^0 + \Phi + q \cdot t_0 + g_q \cdot 2\pi) \qquad (4.2.4)\end{aligned}$$

And thus with (4.2.1e-f)

$$\arg(\tilde{x}_n) = \Phi_n^0 + \Phi(1+\alpha-\beta) + t_0(n+\alpha r-\beta q) + \\ +2\pi(g_n + \alpha g_r - \beta g_q) \qquad (4.2.5)$$

Due to invariance the factors at Φ and t_0 must be deleted:

$$1+\alpha-\beta \overset{!}{=} 0$$
$$n+\alpha r-\beta q \overset{!}{=} 0 \qquad (4.2.6)$$

From that result the conditions denoted in (4,2,1b) for linear factors:

$$\alpha = \frac{q-n}{r-q}$$
$$\beta = \frac{r-n}{r-q} \qquad (4.2.7)$$

which satiesfies the neccessary condition of invariance.

The sufficient condition of completeness demands a unique relation between the invariants $\arg(\tilde{x}_n)$ and the reference phase Φ_n^0

Inserting eq. (4.2.7) in (4.2.4) results in:

$$\arg(\tilde{x}_n) = \Phi_n^0 - \frac{2\pi}{r-q}[(r-q)g_n + (q-n)g_r - (r-n)g_q] \quad (4.2.8)$$

The passing of a pattern into itself when rotated about the balance point with $\Phi = 2\pi/s$, is called a rotation symmetry with degree s . For rotation angle and starting point translation this means equivalence of

$$\begin{aligned} &\Phi \sim \Phi \bmod(2\pi/s), \\ &t_0 \sim t_0 \bmod(2\pi/s) \end{aligned} \quad (4.2.9)$$

and for the Fourier coefficients follows, that only for certain indices their values can be non-zero, namely

$$c_n \equiv 0 \quad \text{for} \quad n \neq 1 + k \cdot s, \qquad k \in \mathbb{Z} \qquad (4.2.10)$$

A general approach for the indices r,q and n in (4.2.8), results in:

$$
\begin{aligned}
n &= 1 + K \cdot s, \\
q &= 1 + K_1 \cdot s, \\
r &= 1 + K_2 \cdot s, \qquad K, K_1, K_2 \in \mathbb{Z}
\end{aligned}
\tag{4.2.11}
$$

Inserting in eq. (4.2.8) results in

$$
\begin{aligned}
\arg(\tilde{x}_n) &= \Phi_n^0 - 2\pi \frac{s}{r-q}\left[(K_2 - K_1) g_n + (K_1 - K) g_r - (K_2 - K) g_q\right] \\
&= \Phi_n^0 + 2\pi \frac{s}{r-q} \cdot g, \qquad g \in \mathbb{Z}
\end{aligned}
\tag{4.2.12}
$$

With the condition

$$r - q = s \tag{4.2.13}$$

a unique correlation between reference and invariants phase can be derived in eq. (4.2.12). Gl. (4.2.12) can be resolved into Φ_n^0 , and thus

$$\Phi_n^0 = (\arg(\tilde{x}_n)) \bmod (2\pi) = \Phi_n + \alpha \cdot \Phi_r - \beta \Phi_q \tag{4.2.14}$$

The minimality of invariants follows from the fact, that for a reconstruction of a pattern of class $\mathbb{F}^N$ generally all Fourier coefficients c_n^0 and thus all invariants are neccessary.

♣

In order to obtain robustness against noise while calculating invariants, preferrably the Fourier coefficients with greatest amplitude are to be chosen for the reference values c_q, c_r .

For disturbed patterns (4.2.10) is not optimally satisfied. As a measure for the rotation symmetry of degree s for disturbed patterns the following sum

$$\sum_{n \neq 1+ks} |c_n|, \qquad k \in \mathbb{Z} \tag{4.2.15}$$

with additional threshold term can be used.

In most cases the i.a. dominant Fourier coefficient c_1 is proper for normalization (q = 1). If the pattern is not rotation symmetric (s = 1), calculation of the Fourier descriptors as proposed in sentence (4.2.1) simplifies to

$$\{\tilde{x}_n := \frac{|c_n|}{|c_1|} e^{j(\Phi_n + (1-n)\Phi_2 - (2-n)\Phi_1)}\} \tag{4.2.16}$$

due to

$$q = 1, \quad s = 1, \quad r = 2,$$

$$\alpha = (1-n), \quad \beta = (2-n) \tag{4.2.1b}$$

Determination of the transformation parameters

In addition to the simple classification in most cases the exact determination of location, orientation, starting point translation and dilation factor is essential. For example, when an industrial robot is picking up things, the fixed reference point x_B can be put at an advantageous point for aimed grabbing.

The unknown values can be retrieved from the values of the Fourier coefficients used for normalization (4.2.1c-f). For ***dilation*** results

$$R = \left| c_q \right| \qquad (4.2.18)$$

rotation angle Φ and starting point translation t_0 result from the argument of c_q and c_r (see 4.1.7), analog for their amplitudes applies eq (4.2.4)

$$\Phi_q = \Phi_q^0 + \Phi + q \cdot t_0 + g_q \cdot 2\pi,$$

$$\Phi_r = \Phi_r^0 + \Phi + r \cdot t_0 + g_r \cdot 2\pi \qquad (4.2.19)$$

Considering the normalization conditions (4.2.1e-f) ($\Phi_q^0 = \Phi_r^0 = 0$) and the side condition r-q=s the unknown ***starting point*** can be calculated by subtraction.

$$t_0 = \left(\frac{\Phi_r - \Phi_q}{r - q} \right) \mathrm{mod}(2\pi/s) \qquad (4.2.20)$$

See eq. (4.2.9).

The ***rotation*** also results from (4.2.19)

$$\Phi = \left(\frac{r\Phi_q - q\Phi_r}{r - q} \right) \mathrm{mod}(2\pi/s) \qquad (4.2.21)$$

From the Fourier coefficient c_0, which denotes the location of the balance point, the ***translation*** can be retrieved

$$z = x_B = c_0 - c_0^0 \cdot R \cdot e^{j\Phi} \qquad (4.2.22)$$

These values refer to the reference pattern, that is normalized in eq. (4.2.1c-f).

Transformation of a original space in a canonical description

Transferring the direct object representation into the instrinsic, invariant parameters “Fourier descriptors” and the transformation or motion parameters “position, size and rotation” :