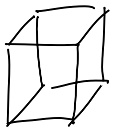

# Kapitel 5c

## Berechnung von Invarianten für diskrete Objekte



# Invariants for Discrete Structures – An Extension of Haar Integrals over Transformation Groups to Dirac Delta Functions

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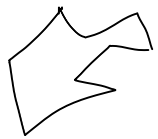
http://lmb.informatik.uni-freiburg.de/



# Summary



1. Introduction
2. Invariants for continuous objects
3. Invariants for discrete objects
   - Invariants for polygons
   - 3D-meshes
   - Discrimination performance and completeness
4. Experiments: Object classification in a Tangram database
5. Conclusions



# Introduction

- Increased interest in 3D models and 3D sensors induce a growing need to support e.g. the automatic search in such databases
- As the description of 3D objects is not canonical ➔ use invariants for their description

# Invariant integration over Euclidean group

For (cyclic) image translation and rotation:

$$(g\mathbf{X})[i,j] = \mathbf{X}[k,l]$$

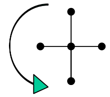
$$\begin{pmatrix} k \\ l \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} i \\ j \end{pmatrix} - \begin{pmatrix} t_0 \\ t_1 \end{pmatrix}$$

all indices to be understood modulo the image dimensions.

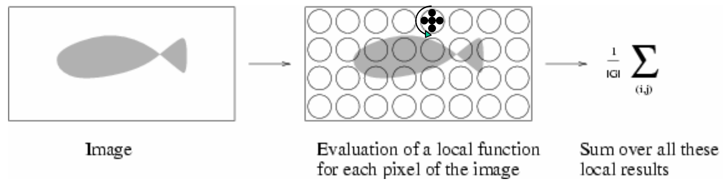
$$A[f](\mathbf{X}) = \frac{1}{2\pi NM} \int_{t_0=0}^{N} \int_{t_1=0}^{M} \int_{\varphi=0}^{2\pi} f(g\mathbf{X})\, d\varphi\, dt_1\, dt_0$$



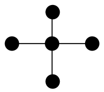
Use as kernel functions monomials of pixels of local support and integrate over the Euclidean motion:

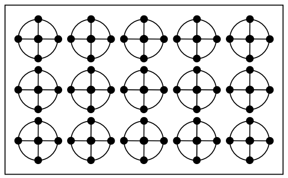


$$f(\mathbf{X}) = m_{00}^3 m_{01}^1 m_{0-1}^5 m_{10}^2 m_{-10}^3$$

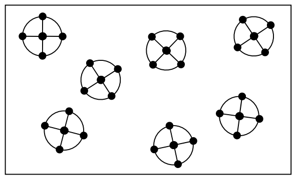






Monomial



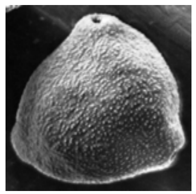
Deterministic integral over the planar Euclidean motion

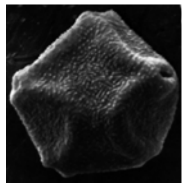


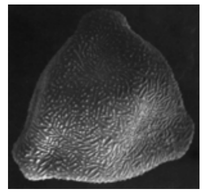
Monte-Carlo-Integration

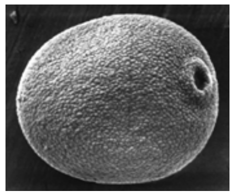


# Pollen examples

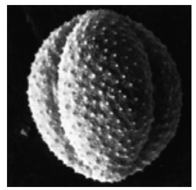

Hasel


Erle


Birke

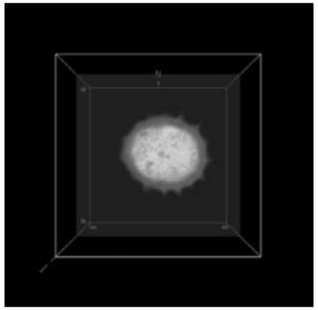

Gräser


Roggen


Beifuß

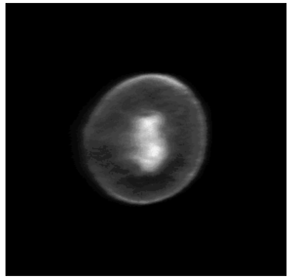
+ 33 further species (not relevant for allergies)

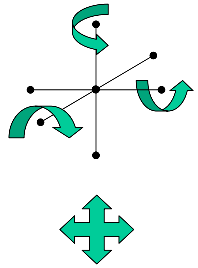
# Gänseblümchen/daisy pollen grain





# Eibe/Taxus





Integrate over Euclidean Motion



# Classification Results using 3D LSM Data

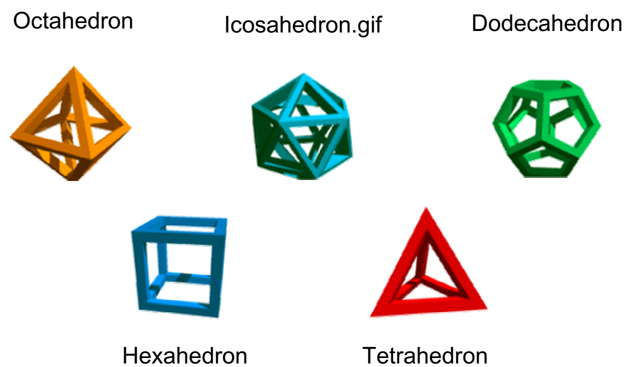
(leave-one-out Classification)

| | Correct | Wrong classifications |
|---|---|---|
| *Artemisia:* | 13 | 1 -> *Compositae*, 1 -> *Platanus* |
| *Alnus:* | 15 | - |
| *Alnus viridis:* | 12 | - |
| *Betula:* | 13 | 2 -> *Plantago* |
| *Corylus:* | 13 | 1 -> *Alnus* |
| *Gramineae/Poaceae:* | 15 | - |
| *Secale:* | 11 | 3 -> *Fagus*, 1 -> *Tilia* |
| Allergolocial irrelevant*: | 282 | 2 -> *Gramineae* |
| Total: | **97.4%** | **2.6%** |

* *Acer, Carpinus, Chenopodium, Compositae, Cruciferae, Fagus, Quercus, Aesculus, Juglans, Fraxinus, Plantago, Platanus, Rumex, Populus, Salix, Taxus, Tilia, Ulmus, Urtica*



# The Five Platonic Solids




A platonic solid is a polyhedron all of whose faces are congruent regular polygons, and where the same number of faces meet at every vertex. The best know example is a *cube* (or *hexahedron* ) whose faces are six congruent squares.

# Extension of Haar-Integrals to Discrete Structures

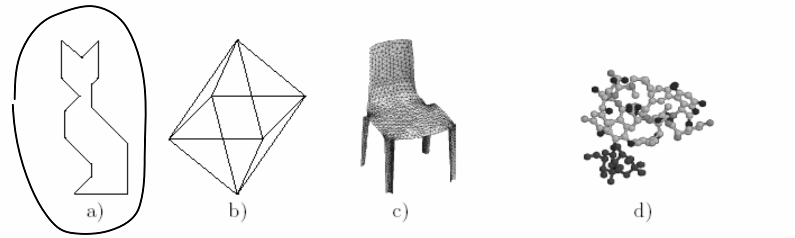



**Fig. 1.** Discrete structures in 2D and 3D: (a) closed contour described by a polygon (b) wireframe object (c) 3D triangulated surface mesh (d) molecule.

Describe discrete structures with Dirac delta functions!



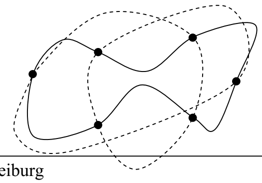
# Invariants on discrete Structures

(topologically equivalent structures)

Chose proper kernel functions on distributions

$$T[f](\mathbf{X}) = \int_G f(g\mathbf{X})dg$$

1. Chose kernel functions which are different from zero only at the vertices and which act only on neighborhoods of degree *m*.
2. As each vertex can be visited in an arbitrary permutation of all points by a continuous Euclidean motion the integral is changed into a invariant sum over all vertices with Euclidean-invariant local discrete features !!
3. Use principle of rigidity to reach completeness: use a basis of features which are locally rigid and which can be pieced together in a unique way to the global object (see invariants for triangle!).





# Invariants for discrete objects

1. For a discrete object Δ and a kernel function *f*(Δ) it is possible to construct an invariant feature *T*[*f*](Δ) by integrating *f*(*g*Δ) over the transformation group $g \in G$.
2. The kernel function is properly designed, such that it delivers a value dependent on the discrete features of a local neighborhood, when a vertex of the object moved by the continuous Euclidean motion *g* hits the origin and has one specific orientation.
3. Let us assume that our discrete object is different from zero only at its vertices. A rotation and translation invariant local discrete kernel function *h* takes care for the algebraic relations to the neighboring vertices and we can write:

$$f(\Delta) = \sum_{i\in\mathbb{V}} h(\Delta, \mathbf{x}_i)\delta(\mathbf{x} - \mathbf{x}_i)$$

where $\mathbb{V}$ is the set of vertices and $\mathbf{x}_i$ the vector representing vertex *i*.

4. In order to get finite values from the distributions it is necessary to introduce under the Haar integral another integration over the spatial domain **X**.
5. By choosing an arbitrary integration path in the continuous group *G* we can visit each vertex in an arbitrary order the integral is transformed into a sum over all local discrete functions allowing all possible permutations of the contributions of the vertices.



# Extension of Haar-Integrals to Discrete Structures

$$T[f](\Delta) := \int_G \int_{\mathbf{X}} f(g\Delta)\,d\mathbf{x}\,dg = \int_G \left[ \int_{\mathbf{X}} \sum_{i\in\mathbb{V}} h(g\Delta, g\mathbf{x}_i)\delta(g\mathbf{x} - g\mathbf{x}_i)\,d\mathbf{x} \right] dg$$

$$= \int_G \left[ \sum_{i\in\mathbb{V}} h(\Delta, \mathbf{x}_i) \right] dg = \sum_{i\in\mathbb{V}} h(\Delta, \mathbf{x}_i)$$

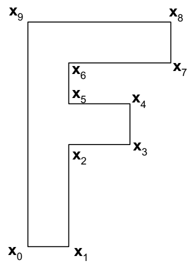
Intuitive result: get *global* Euclidean invariants by summation over discrete *local* Euclidean invariants $h(\boldsymbol{\Delta},\mathbf{x}_i)$ !

Remember: The delta function has the following selection property:

$$\int_{-\infty}^{+\infty} f(x)\delta(x-a)dx = f(a)$$

## Euclidean Invariants for Polygons

We assume e.g. to have given a polygon with 10 vertices, e.g.

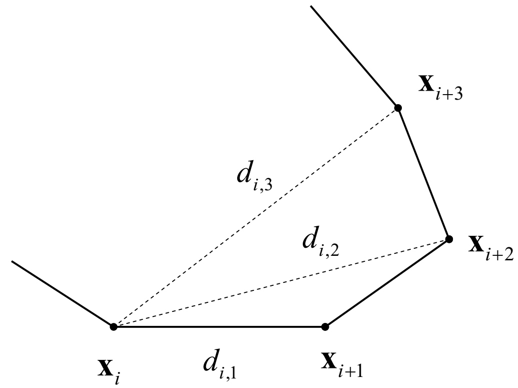



$$\{\mathbf{x}_i\} = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} 0 & 1 & 1 & 2.5 & 2.5 & 1 & 1 & 3.5 & 3.5 & 0 \\ 0 & 0 & 2.5 & 2.5 & 3.5 & 3.5 & 4.5 & 4.5 & 5.5 & 5.5 \end{array}\right]$$



## Choose as local Euclidean invariants distances of vertex i and its k-th righthand neighbours:

$$d_{i,k} = \left\| \mathbf{x}_i - \mathbf{x}_{<i+k>} \right\|$$



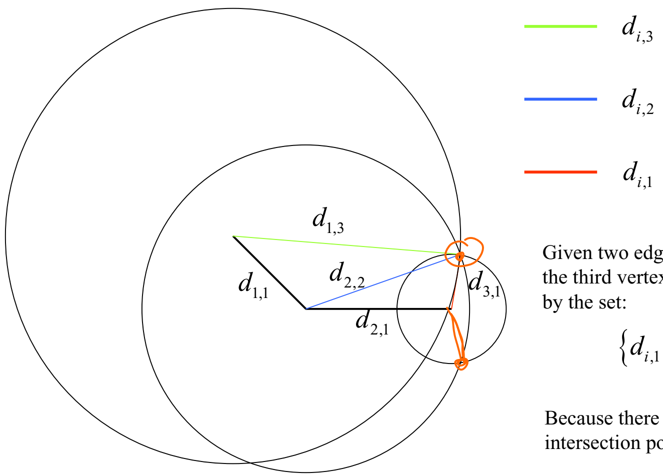

The elements:

$$\{d_{i,1}, d_{i,2}, d_{i,3}\}$$

form a basis for a polygon, because they uniquely define a polygon (up to a mirror-polygon) !

**Principle of rigidity**






Given two edges $d_{1,1}$ and $d_{2,1}$. Then the third vertex is uniquely defined by the set:

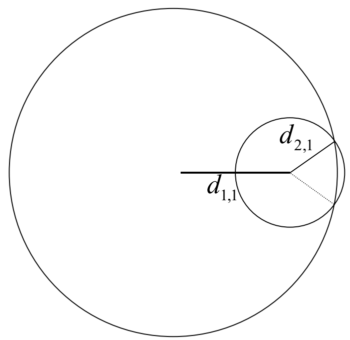
$$\{d_{i,1}, d_{i,2}, d_{i,3}\}$$

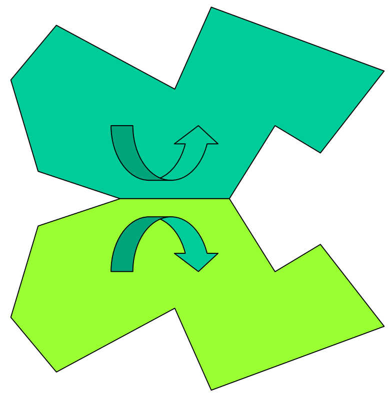
Because there is a unique intersection point of three circles.

This means, that a whole polygon can be uniquely generated by this basis elements iteratively.



With the first two distances we get two initial configurations. Then all further vertices will be unique.
The two initial configurations give two possible polygons, where one is just the mirror image of the other along the first edge as its axis.



As discrete functions of local support we derive *monomials* from distances between neighbouring vertices and hence we get invariants by summing these discrete functions of local support (DFLS) over all vertices:

$$\tilde{x}_{n_1,n_2,n_3,n_4} = \sum_{i\in\mathbb{V}} h(\Delta,\mathbf{x}_i) = \sum_{i\in\mathbb{V}} d_{i,1}^{n_1} d_{i,2}^{n_2} d_{i,3}^{n_3} d_{i,4}^{n_4}$$

Chosing the following 8 values for the exponents we would end up with a corresponding invariant feature vector and a set of 8 invariants:

| $i$ | $n_1$ | $n_2$ | $n_3$ |
|---|---|---|---|
| $\tilde{x}_0$ | 1 | 0 | 0 |
| $\tilde{x}_1$ | 1 | 1 | 0 |
| $\tilde{x}_2$ | 1 | 0 | 1 |
| $\tilde{x}_3$ | 1 | 1 | 1 |
| $\tilde{x}_4$ | 2 | 0 | 0 |
| $\tilde{x}_5$ | 2 | 1 | 0 |
| $\tilde{x}_6$ | 2 | 0 | 1 |
| $\tilde{x}_7$ | 2 | 1 | 1 |

We clearly recognize e.g. $\tilde{x}_0$

as the circumference of the polygon as an invariant.
For the above example of the letter F we get the following invariants:

$$\tilde{\mathbf{x}} = [21 \quad 44 \quad 83.82 \quad 68.6 \quad 184.6 \quad 665.3 \quad 149.5 \quad 751.6]^T$$

**How complete is this set of invariants?**



# Discrimination Performance, question of completeness

We expect a more and more complete feature space by summing over an increasing number of monomials of this basis elements

Looking at a triangle as the most simplest polygon one can show that the following three features derived from the three sides $\{a,b,c\}$ form a complete set of invariants:

$$\tilde{x}_0 = a+b+c, \qquad \tilde{x}_1 = a^2+b^2+c^2, \qquad \tilde{x}_2 = a^3+b^3+c^3 \quad .$$

These features are equivalent to the elementary symmetrical polynomials in 3 variables which are a complete set of invariants with respect to all permutations.



# Completeness for finite Groups

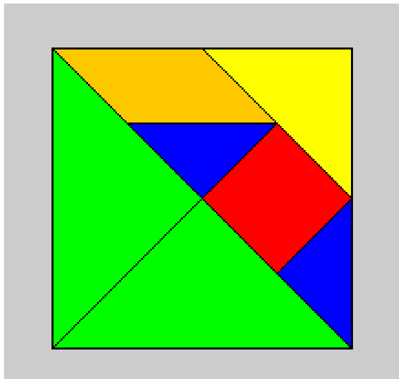
(Emmy Noether, 1916)

For finite Groups $G$ with $|G|$ elements and patterns of dimensionality $N$ the group averages over all monomials of degree $\leq |G|$ are complete and form a basis of the pattern space. The number of monomials is given by

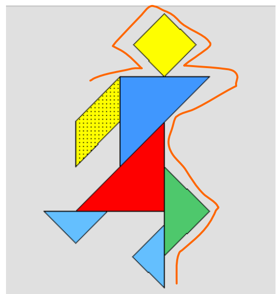
$$\binom{N+|G|}{N}$$



# Experiment: Object classification in a Tangram database

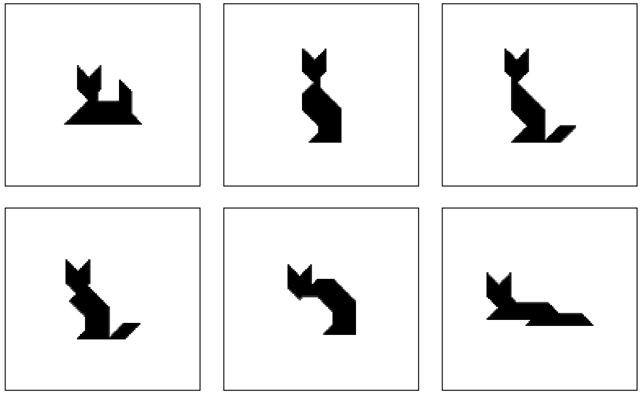


Take the outer contour as feature

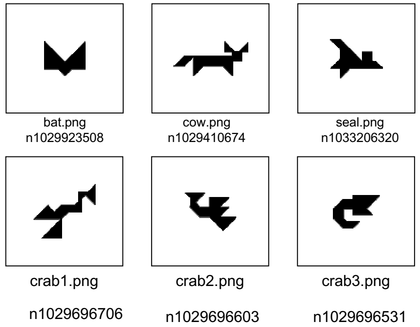


Objects can not easily be discriminated with trivial geometric features!

# Cats



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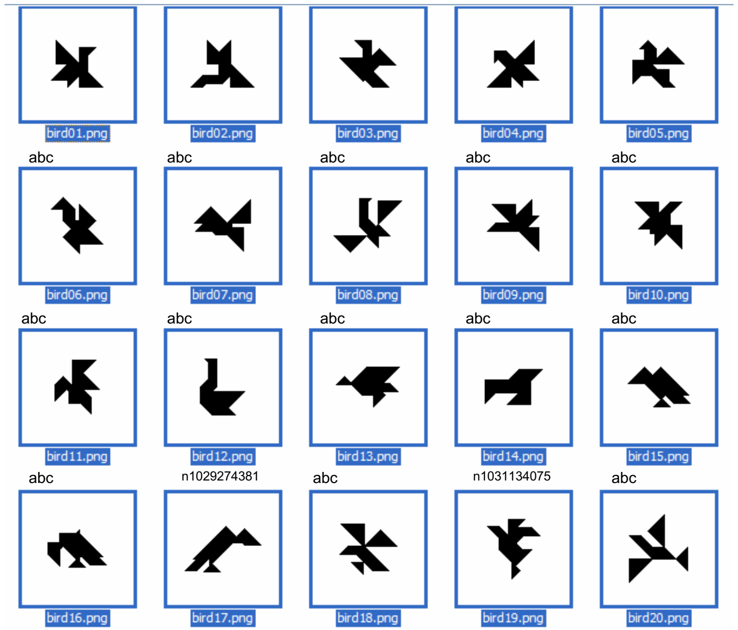
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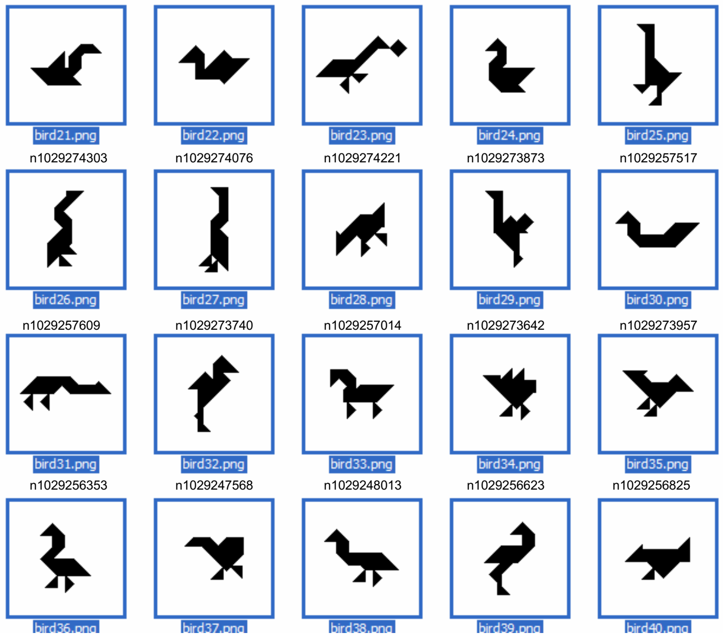
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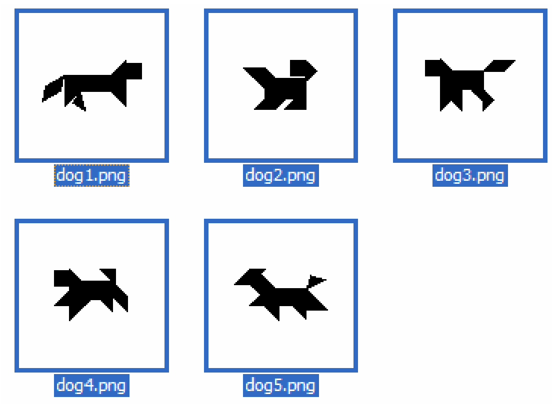
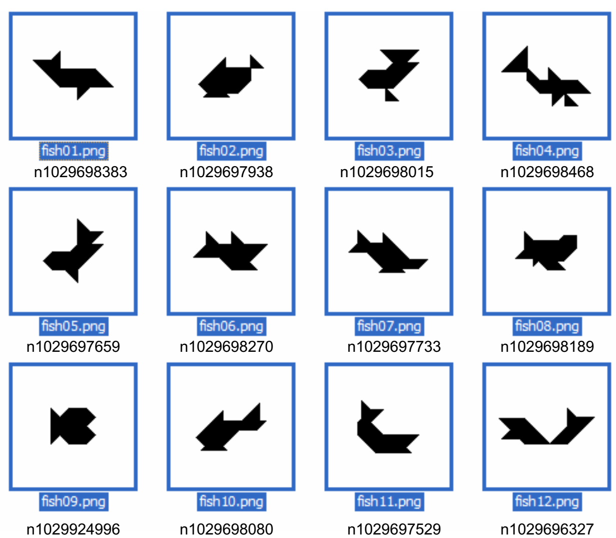
n1029679376 n1029679294 n1029679234

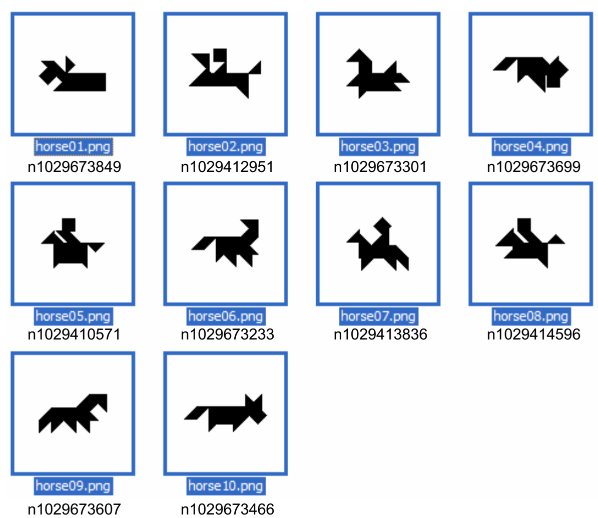


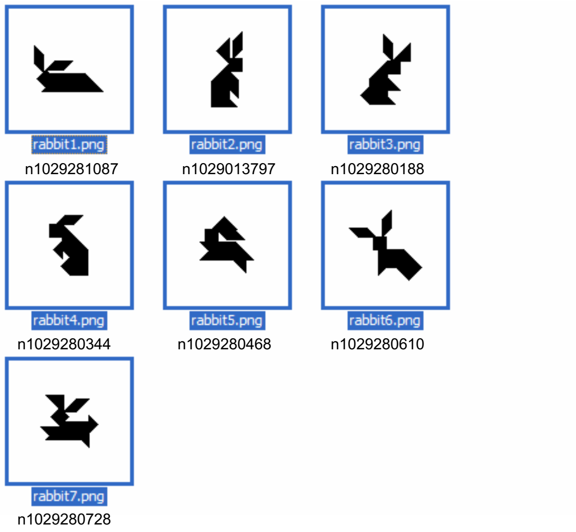

n1029679142 n1029679056



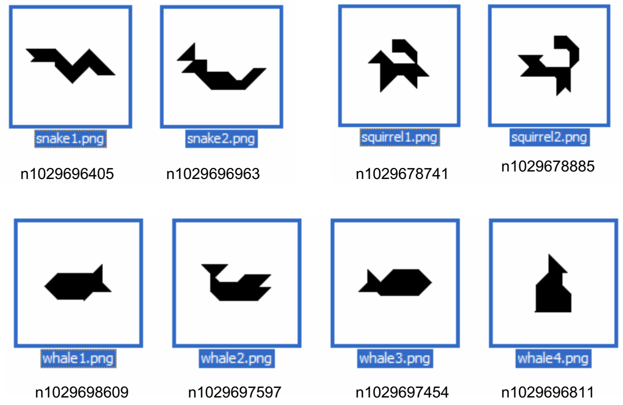










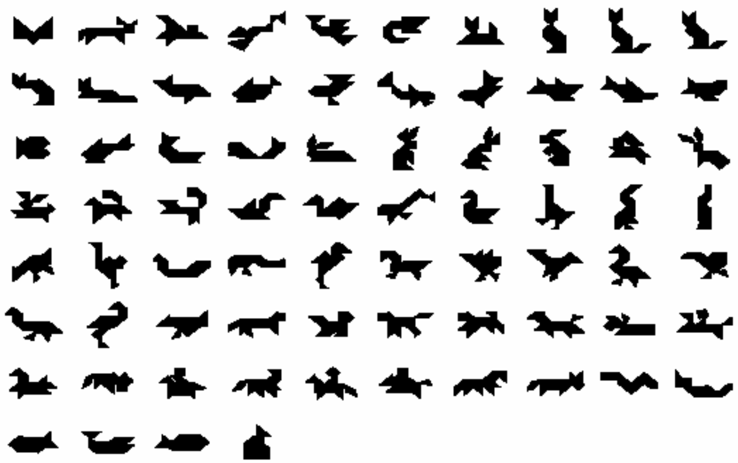



**Fig. 6.** The 74 tangrams used in the experiment.



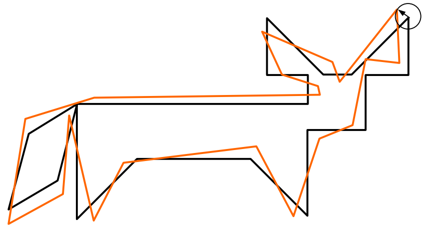
The experiments were conducted with three sets of 6, 10 and 14 invariants respectively using the following exponent table:

| $i$ | $\tilde{x}_0$ | $\tilde{x}_1$ | $\tilde{x}_2$ | $\tilde{x}_3$ | $\tilde{x}_4$ | $\tilde{x}_5$ | $\tilde{x}_6$ | $\tilde{x}_7$ | $\tilde{x}_8$ | $\tilde{x}_9$ | $\tilde{x}_{10}$ | $\tilde{x}_{11}$ | $\tilde{x}_{12}$ | $\tilde{x}_{13}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n_1$ | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 2 | 0 | 0 | 0 |
| $n_2$ | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 2 | 0 | 0 |
| $n_3$ | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 2 | 0 |
| $n_4$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 2 |

$$\tilde{x}_{n_1,n_2,n_3,n_4} = \sum_{i\in\mathbb{V}} h(\Delta, \mathbf{x}_i) = \sum_{i\in\mathbb{V}} d_{i,1}^{n_1} d_{i,2}^{n_2} d_{i,3}^{n_3} d_{i,4}^{n_4}$$

The classification performance was measured against additive noise of 5%, 10% and 20%.

# 10% Noise



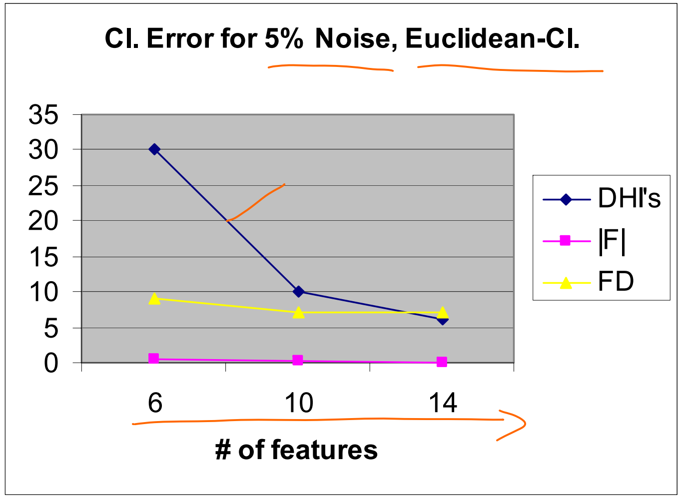


Classification error (in percent) for 5%, 10% and 20% noise for 74 tangrams with a Euclidean (E) and a Mahalanobis (M) Classifier

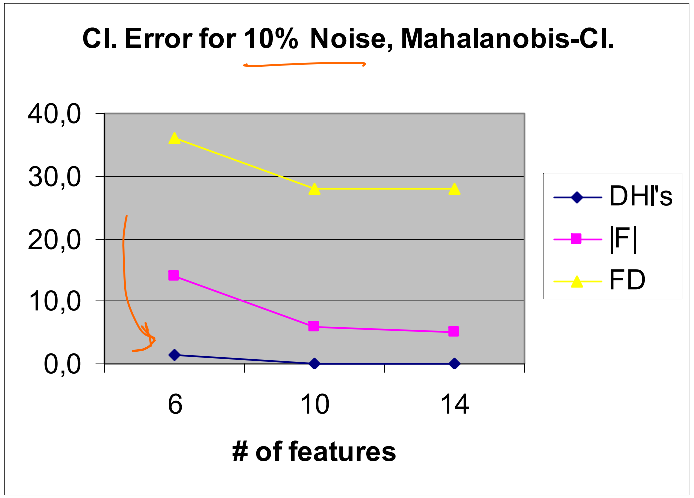
| noise (in percent) | # of invariants | metric | DHI's class. Error in % | FD | \|F\| |
|---|---|---|---|---|---|
| 5 | 6 | E | 30 | 9 | 0.5 |
| 5 | 10 | E | 10 | 7 | 0.2 |
| 5 | 14 | E | 6 | 7 | 0 |
| 10 | 6 | M | 1.5 | 36 | 14 |
| 10 | 10 | M | 0 | 28 | 6 |
| 10 | 14 | M | 0 | 28 | 5 |
| 20 | 6 | M | 25 | 75 | 59 |
| 20 | 10 | M | 7 | 70 | 50 |
| 20 | 14 | M | 3 | 70 | 46 |

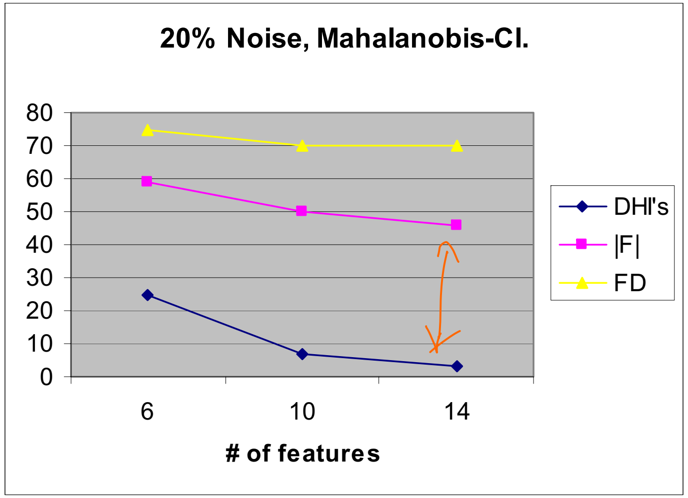
Empirical evaluation for the degree of completeness!





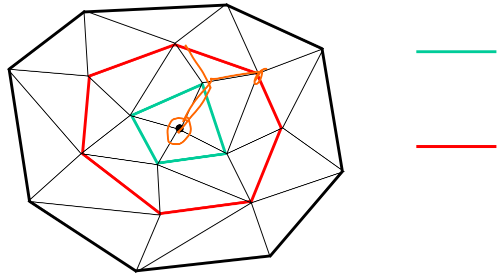


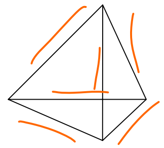








# 3D-Meshes

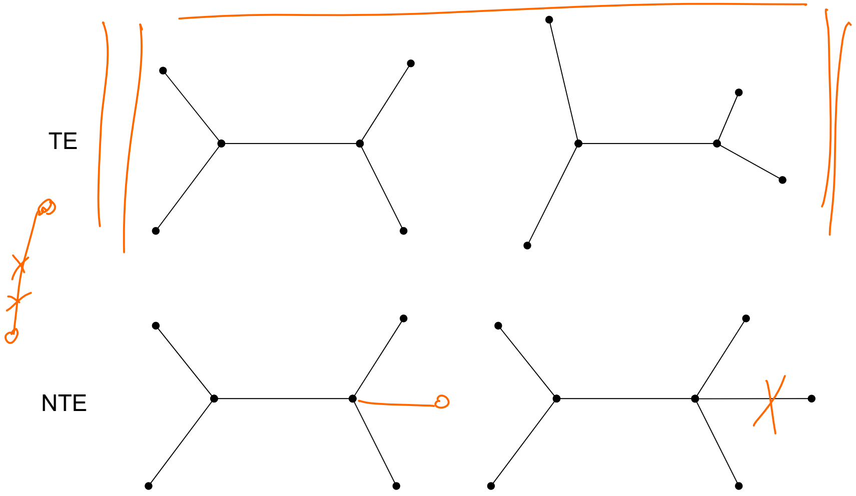


tetrahedron as the basic building block for a polyhedron



## Topologically equivalent structures and (TE) non topologically equivalent structures (NTE)






# Properties

- If we constrain our calculation to a finite number of invariants we end up with a simple *linear complexity* in the number of vertices. This holds if the local neighborhood of vertices is resolved already by the given data structure; otherwise the cost for resolving local neighborhoods must be added.
- In contrast to *graph matching* algorithms we apply here algebraic techniques to solve the problem. This has the advantage that we can apply *hierarchical searches* for retrieval tasks, namely, to start only with one feature and hopefully eliminate already a large number of objects and then continue with an increasing number of features etc.

# Conclusion

- In this paper we have introduced a novel set of invariants for discrete structures in 2D and 3D.
- The construction is a rigorous extension of Haar integrals over transformation groups to Dirac Delta Functions.
- The resulting invariants can easily be calculated with linear complexity in the number of vertices.
- The proposed approach has the potential to be extended to other discrete structures and even to the more general case of weighted graphs.



## Literature: (http://lmb.informatik.uni-freiburg.de)

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