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Efficient Tensor Voting with 3D Tensorial Harmonics

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Abstract

Tensor Voting is an robust technique to extract low-level features in noisy images. The approach achieves its robustness by exploiting coherent orientations in local neighborhoods. In this paper we propose an efficient algorithm for dense Tensor Voting in 3D which makes use of steerable filters. Therefore, we propose steerable expansions of spherical tensor fields in terms of tensorial harmonics, which are its canonical representation. In this way it is possible to perform arbitrary rank tensor voting by linear-combinations of convolutions in an efficient way.

1 Introduction

The Tensor Voting (TV) framework was originally proposed by Medioni et al. [Guy 96] and has found several application in low-level vision in 2D and 3D. For example, it is used for perceptual grouping and extraction of line, curves and surfaces [Mordohai 06]. The key idea is to make unreliable measurements more robust by incorporating neighborhood information in a consistent and coherent manner.

Typically the initial measurements in TV are sparse to provide acceptable running times. Recently, Franken et al. [Franken] proposed an efficient way to compute a dense Tensor Voting in 2D. The idea makes use of a steerable expansion of the voting field. Steerable filters are an efficient architecture to synthesize filters for arbitrary angles from linear combinations of basis filters [Freeman 91]. Perona generalized this concept in [Perona 95] and introduced a methodology to decompose a given filter kernel optimally in a set of steerable basis filters. The idea of Franken et al. [Franken] is to use the steerable decomposition of the voting field to compute the voting process by convolutions in an efficient way. Complex calculus and 2D harmonic analysis are the major mathematical tools that make this approach possible. In this paper we generalize this idea to 3D. Therefore, we introduce so called tensorial harmonics that are the basis for the steerable expansions of spherical tensor fields. They are the 3D generalization of the usual angular Fourier expansion in 2D.

The report is organized as follows: In Section 2 a short introduction to the harmonic analysis of 3D rotations is given. We assume that the reader is familiar with most of the concepts and just give a review and introduce our notations. We also show how spherical and cartesian tensors are related. In Section 3 we propose the tensorial expansion of arbitrary rank spherical tensor fields. In particular, we consider rotational symmetric tensor fields and how they can be steered with respect to 3D rotations. Section 4 applies these principles to TV. The expansion of the voting field can be easily obtained by projections onto a set of orthogonal tensorial harmonics. Finally, we give some examples and hints for implementation and end up with a conclusion.

2 Spherical Tensor Analysis

We will assume that the reader is familiar with the basic notions of the harmonic analysis of $SO(3)$. For introductory reading we recommend mostly literature [Wormer, Rose 95] concerning the quantum theory of the angular momentum, while our representation tries to avoid terms from quantum theory to also give the non-physicists a chance for following. See e.g. [Miller 91, Weinert 80] for introduction from an engineering or mathematical point of view.

In the following we just repeat the basic notions and introduce our notations.

2.1 Preliminaries

Let $\mathbf{D}_g^j$ be the unitary irreducible representation of a $g \in SO(3)$ of order j with $j \in \mathbb{N}$. They are also known as the *Wigner D-matrices* (see e.g. [Rose 95]). The representation $\mathbf{D}_g^j$ acts on a vector space V_j which is represented by $\mathbb{C}^{2j+1}$. We write the elements of V_j in bold face, e.g. $\mathbf{u} \in V_j$ and write the $2j+1$ components in unbold face $u_m \in \mathbb{C}$ where $m = -j, \dots j$. For the transposition of a vector/matrix we write $\mathbf{u}^T$; the joint complex conjugation and transposition is denoted by $\mathbf{u}^\top = \overline{\mathbf{u}}^T$. In this terms the unitarity of $\mathbf{D}_g^j$ is expressed by the formula $(\mathbf{D}_g^j)^\top \mathbf{D}_g^j = \mathbf{I}$.

Note, that we treat the space V_j as a real vector space of dimensions $2j+1$, although the components of $\mathbf{u}$ might be complex. This means that the space V_j is only closed under weighted superpositions with real numbers. As a consequence of this we always have that the components are interrelated by $\overline{u_m} = (-1)^m u_{-m}$. From a computational point of view this is an important issue. Although the vectors are elements of $\mathbb{C}^{2j+1}$ we just have to store just $2j+1$ real numbers.

We denote the standard basis of $\mathbb{C}^{2j+1}$ by $\mathbf{e}_m^j$, where the nth component of $\mathbf{e}_m^j$ is δ_{mn}. In contrast, the standard basis of V_j is written as $\mathbf{c}_m^j = \frac{1+\mathbf{i}}{2}\mathbf{e}_m^j + (-1)^m \frac{1-\mathbf{i}}{2}\mathbf{e}_{-m}^j$. We denote the corresponding 'imaginary' space by $\mathbf{i}V_j$, i.e. elements of $\mathbf{i}V_j$ can be written as $\mathbf{i}\mathbf{v}$ where $\mathbf{v} \in V_j$. So, elements $\mathbf{w} \in \mathbf{i}V_j$ fulfill $\overline{w_m} = (-1)^{m+1} w_{-m}$. Hence, we can write the space $\mathbb{C}^{2j+1}$ as the direct sum of the two spaces $\mathbb{C}^{2j+1} = V_j \oplus \mathbf{i}V_j$. The standard coordinate vector $\mathbf{r} = (x, y, z)^T \in \mathbb{R}^3$ has a natural relation to elements $\mathbf{u} \in V_1$ by

$$\mathbf{u} = \frac{x-y}{\sqrt{2}}\mathbf{c}_1^1 + z\mathbf{c}_0^1 - \frac{x+y}{\sqrt{2}}\mathbf{c}_{-1}^1 = \begin{pmatrix} \frac{1}{\sqrt{2}}(x - \mathbf{i}y) \\ z \\ -\frac{1}{\sqrt{2}}(x + \mathbf{i}y) \end{pmatrix} = \mathbf{S}\mathbf{r} \in V_1$$

Note, that $\mathbf{S}$ is an unitary coordinate transformation. The representation $\mathbf{D}_g^1$ is directly related to the real valued rotation matrix $\mathbf{U}_g \in SO(3) \subset \mathbb{R}^{3\times 3}$ by $\mathbf{D}_g^1 = \mathbf{S}\mathbf{U}_g\mathbf{S}^\top$

Definition 2.1. *A function $\mathbf{f} : \mathbb{R}^3 \mapsto \mathbb{C}^{2j+1}$ is called a spherical tensor field of rank j if it transforms with respect to rotations as*

$$(g\mathbf{f})(\mathbf{r}) := \mathbf{D}_g^j \mathbf{f}(\mathbf{U}_g^T \mathbf{r})$$

for all $g \in SO(3)$. The space of all spherical tensor fields of rank j is denoted by $\mathcal{T}_j$.

2.2 Spherical Tensor Coupling

Now, we define a family of bilinear forms that connect tensors of different ranks.

Definition 2.2. *For every $j \geq 0$ we define a family of bilinear forms of type*

$$\circ_j : V_{j_1} \times V_{j_2} \mapsto \mathbb{C}^{2j+1}$$

where $j_1, j_2 \in \mathbb{N}$ has to be chosen according to the triangle inequality $|j_1 - j_2| \leq j \leq j_1 + j_2$. It is defined by

$$(\mathbf{e}^j_m)^\top(\mathbf{v} \circ_j \mathbf{w}) := \sum_{m=m_1+m_2} \langle jm \mid j_1 m_1, j_2 m_2 \rangle v_{m_1} w_{m_2}$$

where $\langle jm \mid j_1 m_1, j_2 m_2 \rangle$ are the Clebsch-Gordan coefficients.

The characterizing property of these products is that they respect the rotations of the arguments, namely

Proposition 2.3. *Let $\mathbf{v} \in V_{j_1}$ and $\mathbf{w} \in V_{j_2}$, then for any $g \in SO(3)$*

$$(\mathbf{D}^{j_1}_g \mathbf{v}) \circ_j (\mathbf{D}^{j_2}_g \mathbf{w}) = \mathbf{D}^j_g (\mathbf{v} \circ_j \mathbf{w})$$

holds.

Proof. The components of the left-hand side look as

$$\begin{aligned}(\mathbf{e}^j_m)^\top&((\mathbf{D}^{j_1}_g \mathbf{v}) \circ_j (\mathbf{D}^{j_2}_g \mathbf{w})) \\ &= \sum_{\substack{m=m_1+m_2 \\ m'_1 m'_2}} \langle jm | j_1 m_1, j_2 m_2 \rangle D^{j_1}_{m_1 m'_1} D^{j_2}_{m_2 m'_2} v_{m'_1} w_{m'_2}\end{aligned}$$

First, one have to insert the identity by using orthogonality relation (7) with respect to m'_1 and m'_2. Then we can use relation (15) and the definition of $\circ_j$ to prove the assertion. □

Proposition 2.4. *If $j_1 + j_2 + j$ is even, than $\circ$ is symmetric, otherwise antisymmetric. The spaces V_j are closed for the symmetric product, for the antisymmetric product this is not the case.*

$$\begin{aligned} j + j_1 + j_2 \text{ is even } &\Rightarrow \mathbf{v} \circ_j \mathbf{w} \in V_j \\ j + j_1 + j_2 \text{ is odd } &\Rightarrow \mathbf{v} \circ_j \mathbf{w} \in \mathbf{i} V_j, \end{aligned}$$

where $\mathbf{v} \in V_{j_1}$ and $\mathbf{w} \in V_{j_2}$.

Proof. The symmetry and antisymmetry is founded in the symmetry properties of the Clebsch-Gordan coeffcients in equation (13). To show the closure property consider

$$\begin{aligned}
(\mathbf{e}^j_m)^\top \overline{\mathbf{v} \circ_j \mathbf{w}} &= \sum_{m=m_1+m_2} \langle jm|j_1m_1, j_2m_2\rangle \overline{v_{m_1} w_{m_2}} \\
&= \sum_{m=m_1+m_2} (-1)^m \langle jm|j_1m_1, j_2m_2\rangle v_{-m_1} w_{-m_2} \\
&= \sum_{m=m_1+m_2} (-1)^{m+j+j_1+j_2} \langle j(-m)|j_1m_1, j_2m_2\rangle v_{m_1} w_{m_2} \\
&= (-1)^{m+j+j_1+j_2} (\mathbf{e}^j_{-m})^\top \overline{\mathbf{v} \circ_j \mathbf{w}},
\end{aligned}$$

where we used the symmetry property given in equation (14). Hence, we have for even $j + j_1 + j_2$ the 'realness' condition complying to V_j and for odd $j + j_1 + j_2$ the 'imaginaryness' condition for $\mathbf{i}V_j$, which prove the statements. □

For the special case $j = 0$ the arguments have to be of the same rank due to the triangle inequality. Actually in this case the symmetric product coincides with the standard inner product

$$\mathbf{v} \bullet_0 \mathbf{w} = \sum_{m=-j}^{m=j} (-1)^m v_m w_{-m} = \frac{1}{\sqrt{2j+1}} \mathbf{w}^\top \mathbf{v},$$

where j is the rank of $\mathbf{v}$ and $\mathbf{w}$.

The introduced product can also be used to combine tensor fields of different rank by point-wise multiplication.

Proposition 2.5. *Let* $\mathbf{v} \in \mathcal{T}_{j_1}$ *and* $\mathbf{w} \in \mathcal{T}_{j_2}$ *and* j *chosen such that* $|j_1 - j_2| \leq j \leq j_1 + j_2$*, then*

$$\mathbf{f}(\mathbf{r}) = \mathbf{v}(\mathbf{r}) \circ_j \mathbf{w}(\mathbf{r})$$

is in $\mathcal{T}_j$*, i.e. a tensor field of rank* j.

In fact, there is another way to combine two tensor fields: by convolution. The evolving product respects the translation in a different sense.

Proposition 2.6. *Let* $\mathbf{v} \in \mathcal{T}_{j_1}$ *and* $\mathbf{w} \in \mathcal{T}_{j_2}$ *and* j *chosen such that* $|j_1 - j_2| \leq j \leq j_1 + j_2$*, then*

$$(\mathbf{v} \widetilde{\circ}_j \mathbf{w})(\mathbf{r}) := \int_{\mathbb{R}^3} \mathbf{v}(\mathbf{r}' - \mathbf{r}) \circ_j \mathbf{w}(\mathbf{r}')\, d\mathbf{r}'$$

is in $\mathcal{T}_j$*, i.e. a tensor field of rank* j.

2.3 Relation to Cartesian Tensors

The correspondence of spherical and cartesian tensors of rank 0 is trivial. For rank 1 it is just the matrix $\mathbf{S}$ that connects the real-valued vector $\mathbf{r} \in \mathbb{R}^3$ with the spherical coordinate vector $\mathbf{u} = \mathbf{Sr} \in V_1$. For rank 2 the consideration gets more intricate. Consider a real-valued cartesian rank-2 tensor $\mathbf{T} \in \mathbb{R}^{3\times 3}$ and the following unique decomposition

$$\mathbf{T} = \alpha \mathbf{I}_3 + \mathbf{T}_{\text{anti}} + \mathbf{T}_{\text{sym}},$$

where $\alpha \in \mathbb{R}$, $\mathbf{T}_{\text{anti}}$ is an antisymmetric matrix and $\mathbf{T}_{\text{sym}}$ a traceless symmetric matrix. In fact, this decomposition follows the same manner as the spherical tensor decomposition. A rank 0 spherical tensor corresponds to the identity matrix in cartesian notation, while the rank 1 spherical tensor to a antisymmetric 3×3 matrix or, equivalently, to a vector. The rank 2 spherical tensor corresponds to a traceless, symmetric matrix. Let us consider the spherical decomposition. For convenience let $\mathbf{T}^s = \mathbf{STS}^\top$, then the components of the corresponding spherical tensors $\mathbf{b}^j \in V_j$ with $j = 0, 1, 2$ look as

$$b^j_m = \sum_{m_1+m_2=m} \langle 1m_1, 1m_2 | jm \rangle T^s_{m_1 m_2},$$

where $\mathbf{b}^0$ corresponds to α, $\mathbf{b}^1$ to $\mathbf{T}_{\text{anti}}$ and $\mathbf{b}^2$ to $\mathbf{T}_{\text{sym}}$. The inverse of this 'cartesian to spherical'-transformation is

$$T^s_{m_1 m_2} = \sum_{j=0}^{2} \sum_{m=-j}^{m=j} \langle 1m_1, 1m_2 | jm \rangle b^j_m.$$

In particular, consider a cartesian symmetric 2-tensor and its eigensystem. In spherical tensor notation the spherical tensor $\mathbf{b}^2$ is decomposed into products of three 1-tensors $\mathbf{v}_k \in V_1$ as

$$\mathbf{b}^2 = \sum_{k=-1}^{1} \lambda_k \, \mathbf{v}_k \circ_2 \mathbf{v}_k,$$

where $\mathbf{v}_k$ are the eigenvectors of $\mathbf{T}^s$ and λ_k the eigenvalues. Note that $\mathbf{b}^2$ is invariant against a common shift of the eigenvalues by some offset γ. It is 'traceless' in sense that

$$\sum_{k=-1}^{1} \mathbf{v}_k \circ_2 \mathbf{v}_k = \mathbf{0},$$

for any set of orthogonal vectors $\mathbf{v}_{-1}, \mathbf{v}_0, \mathbf{v}_1 \in V_1$. This offset, namely the trace of $\mathbf{T}$ is covered by the zero-rank $\mathbf{b}^0$. It corresponds to the 'ballness' or 'isotropy' of $\mathbf{T}$.

2.4 Spherical Harmonics

We denote the well-known spherical harmonics by $\mathbf{Y}^j : S^2 \to V_j$. We always, write $\mathbf{Y}^j(\mathbf{r})$, where $\mathbf{r}$ may be an element of $\mathbb{R}^3$, but $\mathbf{Y}^j(\mathbf{r})$ is independent of the magnitude of $\mathbf{r}$. We know that the $\mathbf{Y}^j$ provide an orthogonal basis of scalar function on the 2-sphere S^2. Thus, any real scalar field $f \in \mathcal{T}_0$ can be expanded in terms of spherical harmonics in an unique manner:

$$f(\mathbf{r}) = \sum_{j=0}^{\infty} \mathbf{a}^j(r)^\top \mathbf{Y}^j(\mathbf{r})$$

In the following, we always use Racah's normalization (also known as semi-Schmidt normalization), i.e.

$$\langle Y^j_m, Y^{j'}_{m'} \rangle = \frac{1}{4\pi} \int_{S^2} Y^j_m(\mathbf{s}) \, \overline{Y^{j'}_{m'}}(\mathbf{s}) d\mathbf{s} = \frac{1}{2j+1} \delta_{jj'} \delta_{mm'}$$

where the integral ranges over the 2-sphere using the standard measure. One important property of the Racah-normalized spherical harmonics is that $\mathbf{Y}^{j\top} \mathbf{Y}^j = 1$. Another important and useful property is that

$$\mathbf{Y}^j = \frac{1}{\langle j0|j_1 0, j_2 0\rangle} \mathbf{Y}^{j_1} \circ_j \mathbf{Y}^{j_2} \tag{1}$$

if $j + j_1 + j_2$ is even. We can use this formula to iteratively compute higher order $\mathbf{Y}^j$ from given lower order ones. Note that $\mathbf{Y}^0 = 1$ and $\mathbf{Y}^1 = \mathbf{S}\mathbf{r}$, where $\mathbf{r} \in S^2$.

The spherical harmonics have a variety of nice properties. One of the most important ones is that each $\mathbf{Y}^j$, interpreted as a tensor field of rank j is a fix-point with respect to rotations, i.e.

$$(g\mathbf{Y}^j)(\mathbf{r}) = \mathbf{D}^j_g \mathbf{Y}^j(\mathbf{U}^T_g \mathbf{r}) = \mathbf{Y}^j(\mathbf{r})$$

or in other words $\mathbf{Y}^j(\mathbf{U}_g \mathbf{r}) = \mathbf{D}^j_g \mathbf{Y}^j(\mathbf{r})$. A consequence of this is that the expansion coefficients of the rotated function $(gf)(\mathbf{r}) = f(\mathbf{U}^T_g \mathbf{r})$ just look as $\mathbf{D}^j_g \mathbf{a}^j(r)$.

3 Tensorial Harmonic Expansion

We propose to expand a tensor field $\mathbf{f} \in \mathcal{T}_\ell$ of rank ℓ as follows

$$\mathbf{f}(\mathbf{r}) = \sum_{j=0}^{\infty} \sum_{k=-\ell}^{k=\ell} \mathbf{a}_k^j(r) \circ_\ell \mathbf{Y}^j(\mathbf{r}),$$

where $\mathbf{a}_k^j(r) \in \mathcal{T}_{j+k}$ are expansion coefficients. Note, that for $\ell = 0$ the expansion coincides with the ordinary scalar expansion from above. We can further observe that

$$\begin{aligned}(g\mathbf{f})(\mathbf{r}) &= \mathbf{D}_g^\ell \mathbf{f}(\mathbf{U}_g^\top \mathbf{r}) \\ &= \sum_{j=0}^{\infty} \sum_{k=-\ell}^{k=\ell} (\mathbf{D}_g^{j+k} \mathbf{a}_k^j(r)) \circ_\ell \mathbf{Y}^j(\mathbf{r}) \end{aligned} \tag{2}$$

i.e. a rotation of the tensor field affects the expansion coefficients $\mathbf{a}_k^j$ to be transformed by $\mathbf{D}_g^{j+k}$.

By setting $\mathbf{a}_k^j(r) = \sum_{m=-(j+k)}^{m=j+k} a_{km}^j(r) \mathbf{e}_m^{j+k}$ we can identify the functional basis $\mathbf{Z}_{km}^j$ as

$$\mathbf{f}(\mathbf{r}) = \sum_{j=0}^{\infty} \sum_{k=-\ell}^{k=\ell} \sum_{m=-(j+k)}^{m=j+k} a_{km}^j(r) \underbrace{\mathbf{e}_m^{j+k} \circ_\ell \mathbf{Y}^j(\mathbf{r})}_{\mathbf{Z}_{km}^j},$$

Proposition 3.1 (Tensorial Harmonics)**.** *The functions $\mathbf{Z}_{km}^j : S^2 \mapsto V_\ell$ provide an complete and orthogonal basis of the angular part of $\mathcal{T}_\ell$, i.e.*

$$\int_{S^2} (\mathbf{Z}_{km}^j(\mathbf{s}))^\top \mathbf{Z}_{k'm'}^{j'}(\mathbf{s}) d\mathbf{s} = \frac{4\pi}{N_{j,k}} \delta_{j,j'} \delta_{k,k'} \delta_{m,m'},$$

where

$$N_{j,k} = \frac{1}{2\ell+1}(2j+1)(2(j+k)+1).$$

The functions $\mathbf{Z}_{km}^j$ are called the tensorial harmonics.

Proof. We first show the orthogonality by elementary calculations:

$$
\begin{aligned}
&\frac{1}{4\pi}\int_{S^2}(\mathbf{Z}^j_{km}(\mathbf{s}))^\top \mathbf{Z}^{j'}_{k'm'}(\mathbf{s})d\mathbf{s} \\
&= \sum_{M=-\ell}^{\ell} \langle \ell M|(j+k)m, j(M-m)\rangle\langle \ell M|(j'+k')m', j'(M-m')\rangle \underbrace{\frac{1}{4\pi}\int_{S^2}\overline{Y^j_{M-m}}Y^{j'}_{M-m'}}_{\frac{\delta_{j,j'}\delta_{m,m'}}{2j+1}} \\
&= \frac{\delta_{j,j'}\delta_{m,m'}}{2j+1}\underbrace{\sum_{M=-\ell}^{\ell}\langle \ell M|(j+k)m, j(M-m)\rangle\langle \ell M|(j+k')m, j(M-m)\rangle}_{\frac{2\ell+1}{2(j+k)+1}\delta_{(j+k),(j+k')}} \\
&= \delta_{j,j'}\delta_{k,k'}\delta_{m,m'}\frac{1}{2(j+k)+1}\frac{2\ell+1}{2j+1}
\end{aligned}
$$

In line 2 we use the orthogonality of the Racah-normalized spherical harmonics. In the third line we use the orthogonality relation for the Clebsch Gordan coefficients given in (9).

Secondly, we want to show that the expansion of a spherical tensor field $\mathbf{f} \in \mathcal{T}_\ell$ in terms of tensorial harmonics is unique and complete. Everybody agrees that the expansion of the individual components $(\mathbf{e}^\ell_M)^\top\mathbf{f}$ in spherical harmonics is complete. That is, we can write the expansion as

$$
(\mathbf{e}^\ell_M)^\top\mathbf{f}(\mathbf{r}) = \sum_{j=0}^{\infty}\sum_{n=-j}^{j}\mathbf{b}^j_M(r)^\top\mathbf{Y}^j(\mathbf{r}),
$$

where $\mathbf{b}^j_M(r) \in V_j$ are the expansion coefficients for the Mth component. We show the completeness of the tensorial harmonics by connecting them in an one-to-one manner with this ordinary spherical harmonic expansion of the spherical tensor field. For convenience we just consider the jth term in the expansion, i.e. the homogeneous part of $\mathbf{f}$ of order j that we denote by $\mathbf{f}^j$. We start with the expansion in terms of tensorial harmonics and rewrite them to identify the elements of $\mathbf{b}^j_M(r)$ written as $b^j_{M,n}(r)$ in terms of the $a^j_{km}(r)$. And so,

$$\begin{aligned}
(\mathbf{e}^\ell_M)^\top \mathbf{f}^j(\mathbf{r}) &= \sum_{k=-\ell}^{\ell} \sum_{m+n=M} a^j_{km}(r) \langle \ell M | (j+k)m, jn \rangle Y^j_n(\mathbf{r}) \\
&= \sum_{n=-j}^{j} Y^j_n(\mathbf{r}) \underbrace{\sum_{k=-\ell}^{\ell} \sum_{m} a^j_{km}(r) \langle \ell M | (j+k)m, jn \rangle}_{b^j_{M,n}(r)} \\
&= \sum_{n=-j}^{j} b^j_{M,n}(r) Y^j_n(\mathbf{r}).
\end{aligned}$$

Now, we just have to give the inverse relation that computes the a^j_{km} out of the b^j_{Mn}. This can be accomplished by

$$\begin{aligned}
&\sum_{M,n} b^j_{M,n}(r) \langle \ell M | (j+k')m', jn \rangle \\
&\qquad = \sum_{M,n} \sum_{k=-\ell}^{\ell} \sum_{m} a^j_{km}(r) \langle \ell M | (j+k)m, jn \rangle \langle \ell M | (j+k')m', jn \rangle \\
&\qquad = \sum_{k=-\ell}^{\ell} \sum_{m} a^j_{km}(r) \underbrace{\sum_{M,n} \langle \ell M | (j+k)m, jn \rangle \langle \ell M | (j+k')m', jn \rangle}_{\delta_{k,k'} \delta_{m,m'} \frac{2\ell+1}{2(j+k')+1}} \\
&\qquad = \frac{2\ell+1}{2(j+k')+1} a^j_{k'm'}(r),
\end{aligned}$$

where we used again the orthogonality relation for the Clebsch Gordan coefficients given in (9). This provides the one-to-one relation between the tensorial harmonic expansion with the component-wise spherical harmonic expansion and proves the statement. □

3.1 Symmetric Tensor Fields

Typical voting fields used for TV show certain symmetry properties. We figured out three symmetries that let vanish specific terms in the tensorial expansion: the rotationally symmetry with respect to a certain axis, the absence of torsion and reflection symmetry.

The rotation symmetry of a spherical tensor field $\mathbf{f} \in \mathcal{T}_\ell$ about the z-axis is expressed algebraically by the fact that $g_\phi \mathbf{f} = \mathbf{f}$ for all rotation g_ϕ around the z-axis. Such fields can easily be obtained by averaging a general tensor field $\mathbf{f}$ over all these rotations

$$\mathbf{f}_{\text{s}} = \frac{1}{2\pi} \int_0^{2\pi} g_\phi \mathbf{f} \; d\phi.$$

It is well known that the representation $\mathbf{D}^j_{g_\phi}$ of such a rotation is diagonal, namely $D^j_{g_\phi, mm'} = \delta_{mm'} e^{\mathbf{i}m\phi}$. Hence, the expansion coefficients a^j_{km} of $\mathbf{f}_{\text{s}}$ vanish for all $m \neq 0$. Thus, we can write any rotation symmetric tensor field as

$$\mathbf{f}_{\text{s}}(\mathbf{r}) = \sum_{j=0}^{\infty} \sum_{k=-\ell}^{k=\ell} a^j_k(r) \; \mathbf{e}_0^{j+k} \circ_\ell \mathbf{Y}^j(\mathbf{r}). \tag{3}$$

We call such a rotation symmetric field torsion-free if $g_{yz}\mathbf{f}_{\text{s}} = \mathbf{f}_{\text{s}}$, where $g_{yz} \in O(3)$ is a reflection with respect to the yz-plane (or xz-plane). The action of such a reflection on spherical tensors is given by $D^j_{g_{yz}, mm'} = (-1)^m \delta_{m(-m')}$. Similar to the rotational symmetry we can obtain such fields by averaging over the symmetry operation

$$\mathbf{f}_{\text{stf}} = \frac{1}{2}(\mathbf{f}_{\text{s}} + g_{yz}\mathbf{f}_{\text{s}}).$$

Note, that the mirroring operation for a spherical harmonic is just a complex conjugation, that is $\mathbf{Y}^j(\mathbf{U}^T_{g_{yz}}\mathbf{r}) = \overline{\mathbf{Y}^j}(\mathbf{r})$. The consequence for equation (3) is that all terms where the $k + \ell$ are odd vanish. The reason for that is mainly Proposition 2.4 because with its help we can show that

$$\mathbf{D}^\ell_{g_{yz}}(\mathbf{e}_0^{j+k} \circ_\ell \mathbf{Y}^j(\mathbf{U}^T_{g_{yz}}\mathbf{r})) = (-1)^{(k+\ell)}(\mathbf{e}_0^{j+k} \circ_\ell \mathbf{Y}^j(\mathbf{r}))$$

holds.

Finally, consider the reflection symmetry with respect to the xy-plane. This symmetry is particularly important for rank 2 spherical tensor fields. In TV such fields are typically aligned or 'steered' with quantities of the same, even rank. For even rank tensors the parity of the underlying quantity is getting lost, so the voting field has to invariant under such parity changes. This symmetry is algebraically expressed by $g_{xy}\mathbf{f}_s = \mathbf{f}_s$ where $g_{xy} \in O(3)$ is a reflection with respect to the xy-plane, whose action on spherical tensors is given by $D^j_{g_{yz}, mm'} = (-1)^j \delta_{mm'}$. Averaging over this symmetry operation has the consequence that expansion terms with odd j are vanishing. For odd rank tensor fields the reflection symmetry is not imperative. But there is typically an antisymmetry of the form $g_{xy}\mathbf{f}_s = -\mathbf{f}_s$. This antisymmetry let vanish the expansion terms with even index j.

3.2 Expanding Rotation-Symmetric Fields in Polar Representation

We write the spherical tensor field in polar representation $\mathbf{f}(r,\theta,\phi)$, where $\cos(\theta) = z/r$ and $\phi = \arg(x + \mathbf{i}y)$. Consider a field of rank ℓ. In polar representation the rotation symmetry with respect to the z-axis is expressed by the fact that for all $m = -\ell, \ldots, \ell$ we have

$$f_m(r,\theta,\phi) = \alpha_m(r,\theta)e^{\mathbf{i}m\phi},$$

where f_m denote the components of the spherical tensor and $\alpha_m(r,\theta) \in \mathbb{C}$ is colatitudinal/radial dependency of the field. This is easy to see because then $f_m(r,\theta,\phi-\phi')e^{\mathbf{i}m\phi'} = f_m(r,\theta,\phi)$. For torsion-free tensor fields we additionally know that $\alpha_m(r,\theta) \in \mathbb{R}$. To project such a symmetric kind of field on the tensorial harmonics consider the mth component of the tensorial harmonic $\mathbf{Z}^j_{k0}$:

$$\begin{aligned}
(\mathbf{e}^\ell_m)^\top \mathbf{Z}^j_{k0}(\theta,\phi) &= (\mathbf{e}^\ell_m)^\top(\mathbf{e}^{j+k}_0 \circ_\ell \mathbf{Y}^j(\theta,\phi)) \\
&= \langle \ell m \mid (j+k)0, jm\rangle Y^j_m(\theta,\phi) \\
&= \langle \ell m \mid (j+k)0, jm\rangle e^{\mathbf{i}m\phi}\sqrt{\frac{(j-m)!}{(j+m)!}}P^j_m(\cos(\theta)) \\
&= C_{\ell jm}e^{\mathbf{i}m\phi}P^j_m(\cos(\theta))
\end{aligned}$$

Now, using this expression the projection on $\mathbf{Z}^j_{k0}$ yields

$$\begin{aligned}
\langle \mathbf{Z}^j_{k0}, \mathbf{f}\rangle_{S^2} &= \int\limits_{-\pi/2}^{\pi/2}\int\limits_{0}^{2\pi} \mathbf{Z}^j_{k0}(\theta,\phi)^\top \mathbf{f}(r,\theta,\phi)\sin(\theta)\; d\phi d\theta \\
&= 2\pi \sum_{m=-\ell}^{\ell} C_{\ell jm} \int\limits_{-\pi/2}^{\pi/2} \alpha_m(r,\theta)P^j_m(\cos(\theta))\sin(\theta)d\theta
\end{aligned}$$

The residue integral may be computed numerically or analytically.

3.3 Rotational Steering

By equation (2) the tensorial harmonics are very well suited to rotate the expanded spherical tensor field. We want to show how to steer a rotation symmetric field efficiently in a certain direction.

Consider a general rotation $g_{\mathbf{n}} \in SO(3)$ that rotates the z-axis $\mathbf{r}_z = (0, 0, 1)^\top$ to some given orientation $\mathbf{n} \in \mathbb{R}^3$, i.e. $\mathbf{R}_{g_{\mathbf{n}}} \mathbf{r}_z = \mathbf{n}$. Of course, there are several rotations that can accomplish this. But, if we apply such a rotation on a rotational symmetric field $\mathbf{f}_{\mathrm{s}}$ this additional freedom does not have an influence on the result. Starting from the general rotation behavior of the tensorial harmonic expansion in eq. (2) one can derive that the symmetric tensor field $\mathbf{f}_{\mathrm{s}}$ rotates as

$$(g_{\mathbf{n}} \mathbf{f}_{\mathrm{s}})(\mathbf{r}) \;=\; \sum_{j=0}^{\infty} \sum_{k=-\ell}^{k=\ell} a_k^j(r)\, \mathbf{Y}^{j+k}(\mathbf{n}) \circ_\ell \mathbf{Y}^j(\mathbf{r}) \tag{4}$$

This expression is the basis for the proposed method. To prove equation (4) one needs to know that $\mathbf{Y}^j(\mathbf{r}_z) = \mathbf{e}_0^j$.

4 Steerable Tensor Voting

The general idea of Tensor Voting is as follows: Assume, we want to enhance a certain feature in an image, e.g. edges. Therefore, we compute two kind of images. On the one hand a scalar feature image $m : \mathbb{R}^3 \to \mathbb{R}$ that contains the evidence for the occurrence of the feature, e.g. the gradient magnitude in the case of edges. And secondly, an orientational image $\mathbf{n} : \mathbb{R}^3 \to \mathbb{R}^3$ that contains the orientation of the feature of interest, e.g. the gradient direction. Now, the idea is to let each pixel $\mathbf{r}'$ cast tensor-valued votes for the presence of the feature in its neighborhood, where the vote is weighted by the evidence $m(\mathbf{r}')$ for the feature. The orientation of the voting field depends on the local orientation $\mathbf{n}(\mathbf{r}')$. Thus, a position $\mathbf{r}$ gets the contribution $\mathbf{V}^{\mathbf{n}(\mathbf{r}')}(\mathbf{r} - \mathbf{r}')\, m(\mathbf{r}')$ from position $\mathbf{r}'$, where $\mathbf{V}^{\mathbf{n}}$ is the tensor-valued voting field whose superscript determines the direction $\mathbf{n}$ in which the function is oriented. By collecting all contributions from all position $\mathbf{r}'$ in an additive manner we arrive at the final expression for the enhanced feature image

$$\mathbf{U}(\mathbf{r}) = \int_{\mathbb{R}^3} \mathbf{V}^{\mathbf{n}(\mathbf{r}')}(\mathbf{r} - \mathbf{r}')\, m(\mathbf{r}')\, d\mathbf{r}' \tag{5}$$

Now, we restrict to rotational symmetric voting fields. Following the last section we set the voting field to

$$\mathbf{V}^{\mathbf{n}}(\mathbf{r}) = (g_{\mathbf{n}}\mathbf{f}_{\mathrm{s}})(\mathbf{r}),$$

where $\mathbf{f}_{\mathrm{s}}$ is the rotational symmetric field. Inserting this expression in (5) and using eq. (4) yields

$$\begin{aligned}
\mathbf{U}(\mathbf{r}) &= \int_{\mathbb{R}^3} \mathbf{V}^{\mathbf{n}(\mathbf{r}')}(\mathbf{r}-\mathbf{r}')\, m(\mathbf{r}')\, d\mathbf{r}' \\
&= \int_{\mathbb{R}^3} (g_{\mathbf{n}(\mathbf{r}')}\mathbf{f}_{\mathrm{s}})(\mathbf{r}-\mathbf{r}')\, m(\mathbf{r}')\, d\mathbf{r}' \\
&= \left(\begin{array}{l} \displaystyle\int_{\mathbb{R}^3} \sum_{j=0}^{\infty} \sum_{k=-\ell}^{k=\ell} a_k^j(|\mathbf{r}-\mathbf{r}'|) \\ \qquad \mathbf{Y}^{j+k}(\mathbf{n}(\mathbf{r}')) \circ_\ell \mathbf{Y}^j(\mathbf{r}-\mathbf{r}')\, m(\mathbf{r}')\, d\mathbf{r}' \end{array} \right) \\
&= \left(\begin{array}{l} \displaystyle\sum_{j=0}^{\infty} \sum_{k=-\ell}^{k=\ell} \int_{\mathbb{R}^3} \underbrace{m(\mathbf{r}')\mathbf{Y}^{j+k}(\mathbf{n}(\mathbf{r}'))}_{\mathbf{E}^{j+k}(\mathbf{r}')} \circ_\ell \\ \qquad \underbrace{a_k^j(|\mathbf{r}-\mathbf{r}'|)\mathbf{Y}^j(\mathbf{r}-\mathbf{r}')}_{\mathbf{A}_k^j(\mathbf{r}-\mathbf{r}'))}\, d\mathbf{r}' \end{array} \right) \\
&= \sum_{j=0}^{\infty} \sum_{k=-\ell}^{k=\ell} \mathbf{E}^{j+k} \widetilde{\circ}_\ell \mathbf{A}_k^j
\end{aligned}$$

where

$$\mathbf{E}^j(\mathbf{r}) := m(\mathbf{r})\mathbf{Y}^j(\mathbf{n}(\mathbf{r}))$$

are combined tensor-valued evidence images and

$$\mathbf{A}_k^j(\mathbf{r}) := a_k^j(r)\mathbf{Y}^j(\mathbf{r})$$

is the harmonic expansion of the voting field $\mathbf{V}^{\mathbf{r}_z}$ steered in z-direction. The coefficients $a_k^j(r)$ can be obtained by a projection on the tensorial harmonics

$$a_k^j(r) = N_{j,k} \int_{S_r^2} (\mathbf{Z}_{k0}^j(\mathbf{r}))^\top \mathbf{V}^{\mathbf{r}_z}(\mathbf{r})\, d\mathbf{r}, \tag{6}$$

due to the symmetry only $\mathbf{Z}_{k0}^j$ are involved.

Algorithm 1 Voting Algorithm

Input: $m \in \mathcal{T}_0$, $\mathbf{n}(\mathbf{r}) \in \mathcal{T}_1$, $\mathbf{A}^j_k \in \mathcal{T}_j$
Output: $\mathbf{U} \in \mathcal{T}_\ell$
1: Let $\mathbf{E}^0 := m$
2: **for** $j = 1 : (j_{\text{max}} + \ell)$ **do**
3: $\quad \mathbf{E}^j := (\mathbf{E}^{j-1} \circ_j \mathbf{n}) / \langle j0|10, (j-1)0 \rangle$
4: **end for**
5: **for** $j = 0 : j_{\text{max}}$ **do**
6: $\quad$ **for** $k = -\ell : 2 : \ell$ **do**
7: $\quad\quad$ Compute $\mathbf{U} := \mathbf{U} + \mathbf{E}^{j+k} \,\widetilde{\circ}_\ell\, \mathbf{A}^j_k$
8: $\quad$ **end for**
9: **end for**

5 Implementation and Experiments

In Algorithm 1 the voting algorithm is depicted. The input are the evidence image m, an orientation image in spherical notation which is normalized such that $\|\mathbf{n}(\mathbf{r})\| = 1$ and the expansion $\mathbf{A}^j_k$ of the voting field. From line 2-4 the tensor-valued evidence images $\mathbf{E}^j$ are computed iteratively by using equation (1). From line 5-9 the actual voting is performed. The $\widetilde{\circ}_\ell$ operation can performed efficiently by the use of a FFT. The inner loop over k has a step-width of 2 because the voting field is torsion-free. One might use also for outer loop over j a stepwidth of two because of the reflection symmetry with respect to the parity, but in the further experiments we used a stepwidth of one.

To compute the $\mathbf{A}^j_k$ one first have to compute the radius dependent expansion coefficients $a^j_k(r)$ as given in equation (6). In practice, they can be computed analytically or numerically if the analytical way is too difficult. As an example we expanded Medioni's voting field as a rank 1 voting field in a numerical way. Due to the rotation symmetry of the field we sample the voting field just on a 2D polar grid (r, θ), where θ is the colatitudinal angle. The projection onto the tensorial harmonics involves projections onto the associated Legendre polynomial $P^j_0(\cos(\theta))$ and for $m = \pm 1$ on $P^j_{\pm 1}(\cos(\theta))$ and then a weighted sum of the results according to the definition of the tensorial harmonics (see Section 3.2).

In Figure 1 we show approximations for different degrees of expansion. For $j_{\text{max}} = 8$ the artefacts are already very low. We conducted our experiments on a *Intel Xeon X5365 / 3Ghz* (4MB Cache, single threaded). For convolution the *FFTW* is used with 'patient' as planning-mode. In Table 1 we concluded the running

$j_{\max}$	4	6	8	4	6	8
vol	128^3	128^3	128^3	256^3	256^3	256^3
time (s)	5	10	15	45	81	170

Table 1: Computation times for a rank 1 voting field

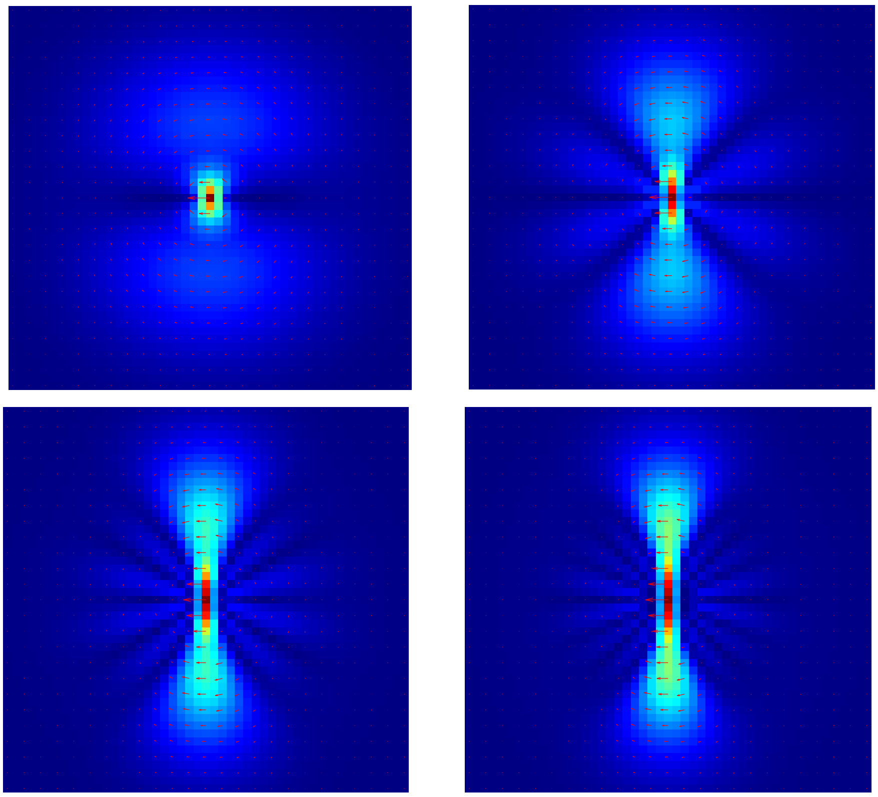

Figure 1: Rank 1 voting field for different degree of expansion ($j_{\max} = 2, 4, 6, 8$)

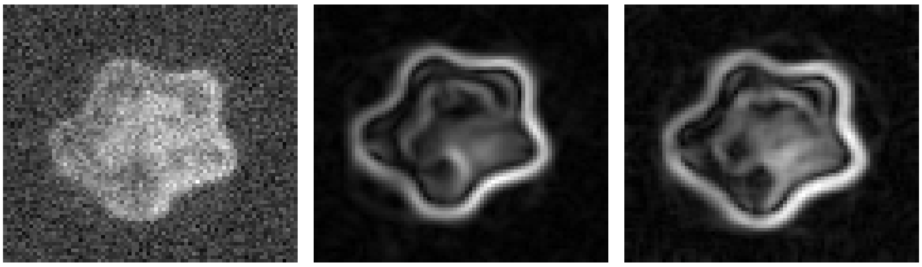

Figure 2: Noisy Confocal Data. On the lower left: after TV processing. On the lower right: Gradient magnitude of smoothed volume ($\sigma = 3px$).

times for different expansion degrees and volume sizes. The times measurements include the computation of the evidence images $\mathbf{E}_j$, their transformation in Fourier domain and the rendering of the voting images. Note, that during rendering we need not to transform back into spatial domain, because just linear operations are involved. Only one FFT at the end is needed to get the final voting result. In Figure 2 we show a toy example for noisy data acquired with confocal laser-scanning microscopy. We applied a rank 1 TV scheme. A slice of the original noisy data together with the magnitude of the TV processed are shown. For comparison, the gradient magnitude of the Gaussian-smoothed image is shown.

6 Conclusion

In this work we have presented an efficient computational scheme for Tensor Voting in 3D. The idea is based on a steerable decomposition of the voting field. Therefore, we proposed so called tensorial harmonics. We firstly presented them in this simple and computationally convenient form. Based on the tensorial expansion of the voting field it is possible to perform the voting process solely by convolutions and spherical multiplications. In toy experiments the validity and speed of the approach were shown.

A Spherical Harmonics

We always use Racah-normalized spherical harmonics. In terms of Legendre polynomials they are written as

$$Y^\ell_m(\phi,\theta) = \sqrt{\frac{(l-m)!}{(l+m)!}} P^\ell_m(\cos(\theta))e^{\mathbf{i}\phi}$$

We always write $\mathbf{r} \in S^2$ instead of (ϕ,θ). The Racah-normalized solid harmonics can be written as

$$R^\ell_m(\mathbf{r}) = \sqrt{(\ell+m)!(\ell-m)!} \sum_{i,j,k} \frac{\delta_{i+j+k,\ell}\delta_{i-j,m}}{i!j!k!2^i2^j}(x-\mathbf{i}y)^j(-x-\mathbf{i}y)^i z^k,$$

where $\mathbf{r} = (x,y,z)$. They are related to spherical harmonics by $R^\ell_m(\mathbf{r})/r^\ell = Y^\ell_m(\mathbf{r})$

B Clebsch Gordan Coefficients

Orthogonality

$$\sum_{j,m} \langle jm|j_1m_1, j_2m_2\rangle\langle jm|j_1m'_1, j_2m'_2\rangle = \delta_{m_1,m'_1}\delta_{m_2,m'_2} \tag{7}$$

$$\sum_{m=m_1+m_2} \langle jm|j_1m_1, j_2m_2\rangle\langle j'm'|j_1m_1, j_2m_2\rangle = \delta_{j,j'}\delta_{m,m'} \tag{8}$$

$$\sum_{m_1,m} \langle jm|j_1m_1, j_2m_2\rangle\langle jm|j_1m_1, j'_2m'_2\rangle = \frac{2j+1}{2j'_2+1}\delta_{j_2,j'_2}\delta_{m_2,m'_2} \tag{9}$$

Special Values

$$\langle \ell m|(\ell-\lambda)(m-\mu), \lambda\mu\rangle = \begin{pmatrix} \ell+m \\ \lambda+\mu \end{pmatrix}^{1/2} \begin{pmatrix} \ell-m \\ \lambda-\mu \end{pmatrix}^{1/2} \begin{pmatrix} 2\ell \\ 2\lambda \end{pmatrix}^{-1/2} \tag{10}$$

$$\begin{aligned}\langle \ell m|(\ell+\lambda)(m-\mu), \lambda\mu\rangle = (-1)^{\lambda+\mu} &\begin{pmatrix} \ell+\lambda-m+\mu \\ \lambda+\mu \end{pmatrix}^{1/2} \\ &\begin{pmatrix} \ell+\lambda+m-\mu \\ \lambda-\mu \end{pmatrix}^{1/2} \begin{pmatrix} 2\ell+2\lambda+1 \\ 2\lambda \end{pmatrix}^{-1/2}\end{aligned} \tag{11}$$

Symmetry

$$\begin{aligned}
\langle jm|j_1m_1, j_2m_2\rangle &= \langle j_1m_1, j_2m_2|jm\rangle & (12)\\
\langle jm|j_1m_1, j_2m_2\rangle &= (-1)^{j+j_1+j_2}\langle jm|j_2m_2, j_1m_1\rangle & (13)\\
\langle jm|j_1m_1, j_2m_2\rangle &= (-1)^{j+j_1+j_2}\langle j(-m)|j_1(-m_1), j_2(-m_2)\rangle & (14)
\end{aligned}$$

C Wigner D-Matrix

The components of $\mathbf{D}^\ell_g$ are written D^ℓ_{mn}. They are called the Wigner D-matrix. In Euler angles ϕ, θ, ψ in ZYZ-convention we have

$$D^\ell_{mn}(\phi, \theta, \psi) = e^{\mathbf{i}m\phi} d^\ell_{mn}(\theta) e^{\mathbf{i}n\psi},$$

where $d^\ell_{mn}(\theta)$ are the Wigner d-matrix which is real-valued. Relation to the Clebsch Gordan coefficients:

$$D^\ell_{mn} = \sum_{\substack{m_1+m_2=m\\ n_1+n_2=n}} D^{\ell_1}_{m_1n_1} D^{\ell_2}_{m_2n_2} \langle lm|l_1m_1, l_2m_2\rangle\langle ln|l_1n_1, l_2n_2\rangle \tag{15}$$

$$D^{\ell_1}_{m_1n_1} D^{\ell_2}_{m_2n_2} = \sum_{l,m,n} D^\ell_{mn} \langle lm|l_1m_1, l_2m_2\rangle\langle ln|l_1n_1, l_2n_2\rangle \tag{16}$$

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