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Spherical Derivatives for Steerable Filtering in 3D

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Abstract

Steerable filters are a valuable tool for various low-level vision tasks. In this paper we address steerable filters for rotations in three dimensions. For two dimensions the theory is well developed. It is common to compute higher order expansions of certain type of filters and steer them according to their maximal response. The simplicity in 2D is due to the Fourier theory: the irreducible subspaces are just one dimensional.

In three dimensions the theory gets much more intricate, the irreducible subspaces grow with each order. This paper proposes a study of the related issues. In particular, we propose a spherical derivative that respects the 3D rotation behavior and connects spherical tensor fields of different degrees. The proposed theory allows to compute steerable filters in a very efficient way by repeated applications of such derivatives. We focus on spherical derivatives of the Gaussian and show how it is possible to construct Gabor filters by the help of these derivatives. We further consider harmonic projections of the Gabor filters, because they are cheaper to compute. They may be considered as canonical surface templates.

1 Introduction

One often needs to compute the same filter responses for different orientations. In the early 1990s Freeman and Adelson [1] introduced an efficient architecture to synthesize filters for arbitrary angles from linear combinations of basis filters. They called their concept 'steerable filters', because it allows one to adaptively steer the filter to any orientation. Nowadays, steerable filters are an indispensable tool in early vision. Perona generalized the concept in [2] to arbitrary deformation groups and introduced a methodology to decompose a given filter kernel optimally in a set of steerable basis filters.

Derivatives of radial symmetric functions play both in 2D and 3D an important role. They naturally provide a steerable set of filters, they are simple to compute and the interpolation functions can be computed analytically. But already in 2D these interpolation functions look already quite cumbersome if the degree of the derivative is higher than two. And also the steering becomes computationally quite costly because all derivatives of a certain order are involved. It was already shown that in 2D there is a solution to this problem by complex analysis [3]. By interpreting the image plane as the complex plane and applying complex derivatives the steering can performed by simple multiplications with a unit complex number. This work is about generalizing this idea to 3D.

We propose two main contributions. On the one hand we introduce so called spherical derivatives. They are characterized by the fact that they behave in a covariant manner with respect to rotations. In this way, they connect spherical tensor fields of different degrees. They replace the usual complex derivatives in the 2D case. Secondly, we work out how any analytical function can be written as a Taylor series in terms of these spherical derivatives. Particularly, we consider Gabor functions in this representation, which have a quite canonical form. Unfortunately the representation of the Gabor still involves spherical derivatives of any order. We will derive that the restriction on a certain subset of coefficients in the spherical expansion of the Gabor is equivalent to projecting the Gabor onto its harmonic part. To see this we compute the reproducing kernel of the space of harmonic functions. In fact, the harmonic projection of the Gabor is just a imaginary evaluation of the reproducing kernel.

Practically, this work is relevant for all those who want to compute steerable responses quickly in 3D in a dense manner. For example, for dense surface detection the proposed harmonic projection of the Gabor is very well suited. The workflow of the algorithm is quite simple. First, apply a smooth onto the 3D image with a Gaussian of width σ determining the scale. Then, apply iteratively the spherical derivative using a finite difference scheme and make a proper weighting of them to determine the shape of the template. The computational effort here is just linear in the number of computed coefficients. The output of the procedure is a number of tensor images of growing dimension as known from usual spherical harmonic representations. Using this representation we can perform a steering just by computing inner products with spherical harmonics. If one wants to find directions of maximal response one can accomplish this by using a 2D FFT for each voxel. Note, that the bottleneck of our approach is the memory consumption. For each voxel in the 3D volume we have to store intermediately $\ell_{\max}^2$ numbers, where $\ell_{\max}$ is the cutoff frequency of the expansion. For large volumes and high $\ell_{\max}^2$ this will cause definitely problems. But this is inherent problem with higher order steerable filters in three dimensions.

A detailed version of this paper can be found in [4], it includes proofs and more detailed explanations. The paper is organized as follows: In the following subsection we make a small review over related work. Then we give in Section 2 a short introduction to the harmonic analysis of 3D rotations. We assume that the reader is familiar with most of the concepts and just give a review and introduce our notations. In Section 3 we introduce the spherical derivative and examine what happens if we apply them multiple times. Using the expansion of the plane wave in spherical harmonics we can derive a Taylor series written in spherical form. This Taylor series also provides a shift formula in differential form. Section 4 introduces briefly the reproducing kernel Hilbert space of harmonic functions and how the reproducing kernel provides an orthogonal projection onto it. We further show how spherical derivatives of Gaussians are directly related to the basis of this space. Then, in Section 5, we show how Gabor functions can be generated by imaginary shifts and how this fact can be used to compute steerable Gabors in an efficient manner. Secondly, we project the Gabor onto its harmonic part, which is much cheaper to compute. Finally, we give some examples and hints for implementation and end up with a conclusion.

1.1 Related Work

While steerable filters are a common tool in image processing and low level computer vision in 2D [2, 5], there are only a few approaches that study 3D steerability. Freeman and Adelson [1] introduced the concept of steerability in 3D and used Gaussian derivatives for filtering. In [6] steerable filtering is used for local orientation analysis and feature detection in 3D. In [7] applications to motion estimation are discussed. Yu et al [8] used conic kernels to improve the orientation resolution.

2 Spherical Tensor Analysis

We will assume that the reader is familiar with the basic notions of the harmonic analysis of $SO(3)$. For introductory reading we recommend mostly literature [9, 10] concerning the quantum theory of the angular momentum, while our representation tries to avoid terms from quantum theory to also give the non-physicists a chance for following. See e.g. [11, 12] for introduction from an engineering viewpoint.

In the following we just repeat the basic notions and introduce our notations. The only uncommon issue in this section is the specific definition of the spherical product.

2.1 Preliminaries

Let $\mathbf{D}^j_g$ be the unitary irreducible representation of a $g \in SO(3)$ of order j with $j \in \mathbb{N}$. They are also known as the *Wigner D-matrices* (see e.g. [10]). The representation $\mathbf{D}^j_g$ acts on a vector space V_j which is represented by $\mathbb{C}^{2j+1}$. We write the elements of V_j in bold face, e.g. $\mathbf{u} \in V_j$ and write the $2j+1$ components in unbold face $u_m \in \mathbb{C}$ where $m = -j, \dots j$. The standard basis of V_j is written as $\mathbf{e}^j_m$. For the transposition of a vector/matrix we write $\mathbf{u}^T$; the joint complex conjugation and transposition is denoted by $\mathbf{u}^\top = \overline{\mathbf{u}}^T$. In this terms the unitarity of $\mathbf{D}^j_g$ is expressed by the formula $(\mathbf{D}^j_g)^\top \mathbf{D}^j_g = \mathbf{I}$.

Note, that we treat the space V_j as a real vector space of dimensions $2j+1$, although the components of $\mathbf{u}$ might be complex. This means that the space V_j is only closed under weighted superpositions with real numbers. As a consequence of this we always have that the components are interrelated by $\overline{u_m} = (-1)^m u_{-m}$. From a computational point of view this is an important issue. Although the vectors are elements of $\mathbb{C}^{2j+1}$ we just have to store just $2j+1$ real numbers. So, the standard coordinate vector $\mathbf{r} = (x, y, z)^T \in \mathbb{R}^3$ has a natural relation to elements $\mathbf{u} \in V_1$ by

$$\mathbf{u} = \begin{pmatrix} \overline{w} \\ z \\ -w \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(x - \mathbf{i}y) \\ z \\ -\frac{1}{\sqrt{2}}(x + \mathbf{i}y) \end{pmatrix} = \mathbf{S}\mathbf{r} \in V_1$$

Note, that $\mathbf{S}$ is an unitary coordinate transformation. Actually, the representation $\mathbf{D}_g^1$ is directly related to the real valued rotation matrix $\mathbf{U}_g \in \mathbb{R}^{3\times 3}$ by $\mathbf{D}_g^1 = \mathbf{S}\mathbf{U}_g\mathbf{S}^\top$

Definition 2.1. *A function $\mathbf{f} : \mathbb{R}^3 \mapsto V_j$ is called a spherical tensor field of rank j if it transforms with respect to rotations as*

$$(g\mathbf{f})(\mathbf{r}) := \mathbf{D}_g^j \mathbf{f}(\mathbf{U}_g^T \mathbf{r})$$

for all $g \in SO(3)$. The space of all spherical tensor fields of rank j is denoted by $\mathcal{T}_j$.

2.2 Spherical Tensor Coupling

Now, we define a family of symmetric bilinear forms that connect tensors of different ranks.

Definition 2.2. *For every $j \geq 0$ we define a family of symmetric bilinear forms of type*

$$\bullet_j : V_{j_1} \times V_{j_2} \mapsto V_j$$

where $j_1, j_2 \in \mathbb{N}$ has to be chosen according to the triangle inequality $|j_1 - j_2| \leq j \leq j_1 + j_2$ and $j_1 + j_2 + j_3$ has to be even. It is defined by

$$(\mathbf{e}_m^j)^\top (\mathbf{v} \bullet_j \mathbf{w}) := \sum_{m=m_1+m_2} \frac{\langle jm \mid j_1 m_1, j_2 m_2 \rangle}{\langle j0 \mid j_1 0, j_2 0 \rangle} v_{m_1} w_{m_2}$$

where $\langle jm \mid j_1 m_1, j_2 m_2 \rangle$ are the Clebsch Gordan coefficients (see e.g. [10]).

Up to the factor $\langle j0 \mid j_1 0, j_2 0 \rangle$ this definition is just the usual spherical tensor coupling equation which is very well known in quantum mechanics of the angular momentum. The additional factor is just for convenience. It normalizes the product such that it shows a more gentle behavior with respect to the spherical harmonics as we will see later.

The characterizing property of these products is that they respect the rotations of the arguments, namely

Proposition 2.3. *Let $\mathbf{v} \in V_{j_1}$ and $\mathbf{w} \in V_{j_2}$, then for any $g \in SO(3)$*

$$(\mathbf{D}_g^{j_1} \mathbf{v}) \bullet_j (\mathbf{D}_g^{j_2} \mathbf{w}) = \mathbf{D}_g^j (\mathbf{v} \bullet_j \mathbf{w})$$

holds.

Proof. The components of the left-hand side look as

$$\begin{aligned}&(\mathbf{e}^j_m)^\top((\mathbf{D}^{j_1}_g \mathbf{v}) \bullet_j (\mathbf{D}^{j_2}_g \mathbf{w}))\\ &\quad = \sum_{\substack{m=m_1+m_2\\ m_1' m_2'}} \frac{\langle jm|j_1m_1,j_2m_2\rangle}{\langle j0 \mid j_1 0, j_2 0\rangle} D^{j_1}_{m_1 m_1'} D^{j_2}_{m_2 m_2'} v_{m_1'} w_{m_2'}\end{aligned}$$

First one have to insert the identity by using orthogonality relation (11) with respect to m_1' and m_2'. Then we can use relation (19) and the definition of $\bullet_j$ to prove the assertion. ☐

For the special case $j = 0$ the arguments have to be of the same rank due to the triangle inequality. Actually, in this case the new product coincides with the standard inner product

$$\mathbf{v} \bullet_0 \mathbf{w} = \sum_{m=-j}^{m=j} (-1)^m v_m w_{-m} = \mathbf{w}^\top \mathbf{v},$$

where j is the rank of $\mathbf{v}$ and $\mathbf{w}$. Further note, that if one of the arguments of $\bullet$ is a scalar, then $\bullet$ reduces to the standard scalar multiplication, i.e. $v \bullet_j \mathbf{w} = v\mathbf{w}$, where $v \in V_0$ and $\mathbf{w} \in V_j$. Another remark is that $\bullet$ is not associative.

The introduced product can also be used to combine tensor fields of different rank by point-wise multiplication.

Proposition 2.4. *Let $\mathbf{v} \in \mathcal{T}_{j_1}$ and $\mathbf{w} \in \mathcal{T}_{j_2}$ and j chosen such that $|j_1 - j_2| \leq j \leq j_1 + j_2$, then*

$$\mathbf{f}(\mathbf{r}) = \mathbf{v}(\mathbf{r}) \bullet_j \mathbf{w}(\mathbf{r})$$

is in $\mathcal{T}_j$, i.e. a tensor field of rank j.

2.3 Spherical Harmonics

We denote the well-known spherical harmonics by $\mathbf{Y}^j : S^2 \to V_j$. We always, write $\mathbf{Y}^j(\mathbf{r})$, where $\mathbf{r}$ my be an element of $\mathbb{R}^3$, but $\mathbf{Y}^j(\mathbf{r})$ is independent of the magnitude of $\mathbf{r}$. We know that the $\mathbf{Y}^j$ provide an orthogonal basis of scalar function on the 2-sphere S^2. Thus, any real scalar field $f \in \mathcal{T}_0$ can be expanded in terms of spherical harmonics in an unique manner:

$$f(\mathbf{r}) = \sum_{j=0}^{\infty} \mathbf{a}^j(r)^\top \mathbf{Y}^j(\mathbf{r})$$

In the following, we always use Racah's normalization (also known as semi-Schmidt normalization), i.e.

$$\langle Y^j_m, Y^{j'}_{m'}\rangle = \frac{1}{4\pi}\int_{S^2} Y^j_m(\mathbf{s})\,\overline{Y^{j'}_{m'}}(\mathbf{s})d\mathbf{s} = \frac{1}{2j+1}\delta_{jj'}\delta_{mm'}$$

where the integral ranges over the 2-sphere using the standard measure. One important property of the Racah-normalized spherical harmonics is that ${\mathbf{Y}^j}^\top\mathbf{Y}^j = 1$. Another important and useful property is that $\mathbf{Y}^j = \mathbf{Y}^{j_1} \bullet_j \mathbf{Y}^{j_2}$. We can use this formula to iteratively compute higher order $\mathbf{Y}^j$ from given lower order ones. Note that $\mathbf{Y}^0 = 1$ and $\mathbf{Y}^1 = \mathbf{S}\mathbf{r}$, where $\mathbf{r} \in S^2$.

The spherical harmonics have a variety of nice properties. One of the most important ones is that each $\mathbf{Y}^j$, interpreted as a tensor field of rank j is a fix-point with respect to rotations, i.e.

$$(g\mathbf{Y}^j)(\mathbf{r}) = \mathbf{D}^j_g\mathbf{Y}^j(\mathbf{U}^T_g\mathbf{r}) = \mathbf{Y}^j(\mathbf{r})$$

or in other words $\mathbf{Y}^j(\mathbf{U}_g\mathbf{r}) = \mathbf{D}^j_g\mathbf{Y}^j(\mathbf{r})$. A consequence of this is that the expansion coefficients of the rotated function $(gf)(\mathbf{r}) = f(\mathbf{U}^T_g\mathbf{r})$ just look as $\mathbf{D}^j_g\mathbf{a}^j(r)$.

Later we will need the following definition

$$\mathbf{R}^n_i(\mathbf{r}) := r^{n+i}\mathbf{Y}^{n-i}(\mathbf{r}).$$

The functions $\mathbf{R}^n := \mathbf{R}^n_0$ are usually called regular solid harmonics and are solutions of the Laplace equation. Note that the $\mathbf{R}^n$ are homogeneous polynomials of order n, meaning $\mathbf{R}^n(\lambda\mathbf{r}) = \lambda^n\mathbf{R}^n(\mathbf{r})$ for any $\lambda \in \mathbb{R}$.

3 Spherical Derivatives

This section proposes the basic tools for dealing with derivatives in the context of spherical tensor analysis. First, we will introduce the spherical derivative which connects spherical tensor fields of different ranks by differentiation. Then, we consider the representations of these derivatives in the Fourier domain. Based on this knowledge our goal is to find a Taylor series expansion in terms of the spherical derivatives. Therefore, we compute the spherical expansion of a plane wave and use the representations of the spherical derivatives in the Fourier domain to obtain a differential shift operator analog to the ordinary cartesian $\tau = e^{\mathbf{t}^T\nabla}$.

Now, let us start with the definition of the spherical derivative.

Proposition 3.1 (Spherical Derivatives)**.** *Let* $\mathbf{f} \in \mathcal{T}_\ell$ *be a tensor field. The spherical up-derivative* $\boldsymbol{\nabla}^1 : \mathcal{T}_\ell \to \mathcal{T}_{\ell+1}$ *and the down-derivative* $\boldsymbol{\nabla}_1 : \mathcal{T}_\ell \to \mathcal{T}_{\ell-1}$ *are defined as*

$$\begin{aligned} \boldsymbol{\nabla}^1 \mathbf{f} &:= \nabla \bullet_{\ell+1} \mathbf{f} & (1) \\ \boldsymbol{\nabla}_1 \mathbf{f} &:= \nabla \bullet_{\ell-1} \mathbf{f}, & (2) \end{aligned}$$

where

$$\nabla = \left(\frac{1}{\sqrt{2}}(\partial_x - \mathbf{i}\partial_y), \partial_z, -\frac{1}{\sqrt{2}}(\partial_x + \mathbf{i}\partial_y)\right)$$

is the spherical gradient and $\partial_x, \partial_y, \partial_z$ *the standard partial derivatives.*

Proof. We have to show that $\boldsymbol{\nabla}^1 \mathbf{f} \in \mathcal{T}_{\ell+1}$, i.e.

$$\boldsymbol{\nabla}^1(\mathbf{D}_g^\ell \mathbf{f}(\mathbf{U}_g^T \mathbf{r})) = \mathbf{D}_g^{\ell+1}(\boldsymbol{\nabla}^1 \mathbf{f})(\mathbf{U}_g^T \mathbf{r})$$

and $\boldsymbol{\nabla}_1 \mathbf{f} \in \mathcal{T}_{\ell-1}$

$$\boldsymbol{\nabla}_1(\mathbf{D}_g^\ell \mathbf{f}(\mathbf{U}_g^T \mathbf{r})) = \mathbf{D}_g^{\ell-1}(\boldsymbol{\nabla}_1 \mathbf{f})(\mathbf{U}_g^T \mathbf{r})$$

Both statements are proved just by using the properties of $\bullet$. ☐

Note, that for a scalar function the spherical up-derivative is just the spherical gradient, i.e. $\nabla f = \boldsymbol{\nabla}^1 f$.

In the Fourier domain the spherical derivatives act by point-wise $\bullet$-multiplications with a solid harmonic $\mathbf{i}k\mathbf{Y}^1(\mathbf{k}) = \mathbf{i}\mathbf{R}^1(\mathbf{k}) = \mathbf{i}\mathbf{S}\mathbf{k}$ where $k = \|\mathbf{k}\|$ the frequency magnitude:

Proposition 3.2 (Fourier Representation)**.** *Let* $\widetilde{\mathbf{f}}(\mathbf{k})$ *be the Fourier transformation of some* $\mathbf{f} \in \mathcal{T}_\ell$ *and* $\widetilde{\boldsymbol{\nabla}}$ *representations of the spherical derivative in the Fourier domain that are implicitly defined by* $\widetilde{(\boldsymbol{\nabla}\mathbf{f})} = \widetilde{\boldsymbol{\nabla}}\widetilde{\mathbf{f}}$*, then*

$$\begin{aligned} \widetilde{\boldsymbol{\nabla}}^1 \widetilde{\mathbf{f}}(\mathbf{k}) &= \mathbf{i}\mathbf{R}^1(\mathbf{k}) \bullet_{\ell+1} \widetilde{\mathbf{f}}(\mathbf{k}) & (3) \\ \widetilde{\boldsymbol{\nabla}}_1 \widetilde{\mathbf{f}}(\mathbf{k}) &= \mathbf{i}\mathbf{R}^1(\mathbf{k}) \bullet_{\ell-1} \widetilde{\mathbf{f}}(\mathbf{k}). & (4) \end{aligned}$$

Proof. By the ordinary Fourier corresponce for the partial derivative, namely $\widetilde{\partial_x \mathbf{f}} = \mathbf{i}k_x \widetilde{\mathbf{f}}$, we can also verify for the spherical gradient ∇ that

$$\widetilde{\nabla} = \mathbf{i}\mathbf{S}\mathbf{k} = \mathbf{i}\mathbf{R}^1(\mathbf{k})$$

and hence

$$\widetilde{\boldsymbol{\nabla}^1 \mathbf{f}} = \widetilde{(\nabla \bullet_{\ell+1} \mathbf{f})} = \widetilde{\nabla} \bullet_{\ell+1} \widetilde{\mathbf{f}} = \mathbf{i}\mathbf{R}^1(\mathbf{k}) \bullet_{\ell+1} \widetilde{\mathbf{f}}$$

which was to show. ☐

Both statements are direct consequences of the Fourier correspondences for the ordinary partial derivatives. For scalar fields we can generalize this statement also for higher orders

Proposition 3.3 (Multiple Spherical Derivatives). *For $n \geq i$ we define $\boldsymbol{\nabla}_i^n : \mathcal{T}_0 \to \mathcal{T}_{n-i}$ by*

$$\boldsymbol{\nabla}_i^n := \boldsymbol{\nabla}_i \boldsymbol{\nabla}^n := \underbrace{\boldsymbol{\nabla}_1 \dots \boldsymbol{\nabla}_1}_{i-times} \underbrace{\boldsymbol{\nabla}^1 \dots \boldsymbol{\nabla}^1}_{n-times}.$$

In the Fourier domain these multiple derivatives act by

$$(\widetilde{\boldsymbol{\nabla}}_i^n \widetilde{f})(\mathbf{k}) \quad = \quad (\mathbf{i})^{n+i}\, \mathbf{R}_i^n(\mathbf{k}) \widetilde{f}(\mathbf{k}). \tag{5}$$

Using this one can show that $\boldsymbol{\nabla}_i^n = \boldsymbol{\nabla}^{n-i}\Delta^i$, where Δ is the Laplace operator.

Proof. Based on the correspondences stated in equations (3) and (4), the basic reason for equation (5) is that $\mathbf{Y}^j = \mathbf{Y}^{j_1} \bullet_j \mathbf{Y}^{j_2}$ and its implications for $\mathbf{R}_i^n$. To see the assertion for $i = 0$ consider the following

$$(\widetilde{\boldsymbol{\nabla}}^n \widetilde{f})(\mathbf{k}) \quad = \quad \mathbf{i}\mathbf{R}^1(\mathbf{k}) \bullet_n (\dots (\mathbf{i}\mathbf{R}^1(\mathbf{k}) \bullet_3 (\mathbf{i}\mathbf{R}^1(\mathbf{k}) \bullet_2 \mathbf{i}\mathbf{R}^1(\mathbf{k}))) \dots) \widetilde{f}(\mathbf{k}).$$

By successively applying $\mathbf{R}^{n+1} = \mathbf{R}^1 \bullet_{n+1} \mathbf{R}^n$ we get immediately $(\widetilde{\boldsymbol{\nabla}}^n \widetilde{f})(\mathbf{k}) = \mathbf{i}^n \mathbf{R}^n(\mathbf{k}) \widetilde{f}(\mathbf{k})$. For $\widetilde{\boldsymbol{\nabla}}_i^n$ it is just the same reasoning but we have to apply successively

$$\mathbf{R}^1 \bullet_{n-i-1} \mathbf{R}_i^n = \mathbf{R}_{i+1}^n.$$

And finally, we prove $\boldsymbol{\nabla}_i^n = \boldsymbol{\nabla}^{n-i}\Delta^i$ in Fourier domain by

$$\begin{aligned} \widetilde{\boldsymbol{\nabla}}_i^n &= \mathbf{i}^{n+i}\mathbf{R}_i^n(\mathbf{k}) = k^{n+i}\mathbf{Y}^{n-i}(\mathbf{k}) \\ &= (\mathbf{i}k)^{n-i}\mathbf{Y}^{n-i}(\mathbf{k})(\mathbf{i}k)^{2i} = \mathbf{i}^{n-i}\mathbf{R}^{n-i}(\mathbf{k})(-k^2)^i \\ &= \widetilde{\boldsymbol{\nabla}}^{n-i}\widetilde{\Delta}^i, \end{aligned}$$

where we used the well known Fourier correspondence of the Laplace operator, namely $\widetilde{\Delta} = -k^2$. $\square$

We want to emphasize that both statements only hold for scalar-valued fields, generalizations to tensor-valued fields are not straight-forward.

3.1 Plane Wave Expansion and Shifts

The expansion of a plane wave in spherical harmonics is known (see e.g [10] p. 136) to be

$$\begin{aligned} e^{\mathbf{i}\mathbf{k}^\top \mathbf{r}} &= \sum_\ell (2\ell+1)(\mathbf{i})^\ell \, j_\ell(kr) \, P_\ell(\tfrac{\mathbf{k}^\top \mathbf{r}}{kr}) \\ &= \sum_\ell (2\ell+1)(\mathbf{i})^\ell \, j_\ell(kr) \, \mathbf{Y}^\ell(\mathbf{r}) \bullet_0 \mathbf{Y}^\ell(\mathbf{k}) \end{aligned}$$

where j_ℓ are the spherical Bessel function of the first kind and P_ℓ the Legendre polynomials. We can also write the plane wave expansion in terms of solid harmonics $\mathbf{R}^n_i$ as follows

Proposition 3.4 (Plane Wave Expansion)**.** *The plane wave expansion can be written as*

$$e^{\mathbf{i}\mathbf{k}^\top \mathbf{r}} = \sum_{n \geq i} \mathbf{i}^{n+i} \alpha_{n,i} \, \mathbf{R}^n_i(\mathbf{r}) \bullet_0 \mathbf{R}^n_i(\mathbf{k}).$$

with

$$\alpha_{n,i} = \frac{(2(n-i)+1)}{i!2^i(2n+1)!!}.$$

Proof. Starting with the usual expression for the plane wave and inserting the Taylor series expansion of the spherical Bessel function (21) gives

$$\begin{aligned} e^{\mathbf{i}\mathbf{k}^\top \mathbf{r}} &= \sum_\ell (2\ell+1)(\mathbf{i})^\ell \, j_\ell(kr) \, \mathbf{Y}^\ell(\mathbf{r}) \bullet_0 \mathbf{Y}^\ell(\mathbf{k}) \\ &= \sum_{\ell,i} \mathbf{i}^\ell \frac{(-1)^i(2\ell+1)}{2^i i!(2(i+\ell)+1)!!} (kr)^{2i+\ell} \mathbf{Y}^\ell(\mathbf{r}) \bullet_0 \mathbf{Y}^\ell(\mathbf{k}) \end{aligned}$$

Now, reindexing by $n = i + \ell$, i.e. replacing any ℓ by $n - i$ yields

$$e^{\mathbf{i}\mathbf{k}^\top \mathbf{r}} = \sum_{n \geq i} \mathbf{i}^{n-i} \frac{(-1)^i(2(n-i)+1)}{2^i i!(2n+1)!!} (kr)^{n+i} \mathbf{Y}^\ell(\mathbf{r}) \bullet_0 \mathbf{Y}^\ell(\mathbf{k}).$$

Further recognizing that $\mathbf{i}^{n-i}(-1)^i = \mathbf{i}^{n+i}$ proves the assertion. □

By $(2n+1)!!$ we denote the double factorial that is defined as $(2n+1)!! = (2n+1)(2n-1)\dots 1$. In Fourier domain the above expression can be used to model a shift by means of a spherical expansion. Consider a scalar function $\widetilde{f}(\mathbf{k})$

and multiply it by $e^{\mathbf{i}\mathbf{k}^\top \mathbf{t}}$, i.e. shift the function in spatial domain, and then using (5) gives

$$
\begin{aligned}
e^{\mathbf{i}\mathbf{k}^\top \mathbf{t}} \widetilde{f}(\mathbf{k}) &= \sum_{n\geq i} \alpha_{n,i}\, \mathbf{R}_i^n(\mathbf{t}) \bullet_0 \left((\mathbf{i}k)^{n+i} \mathbf{Y}^{n-i}(\mathbf{k}) \widetilde{f}(\mathbf{k}) \right) \\
&= \sum_{n\geq i} \alpha_{n,i}\, \mathbf{R}_i^n(\mathbf{t}) \bullet_0 (\widetilde{\boldsymbol{\nabla}}_i^n \widetilde{f}(\mathbf{k})).
\end{aligned}
$$

Transferring this equation to spatial domain the shift can be expressed as

$$
(\tau f)(\mathbf{r}) = f(\mathbf{r} + \mathbf{t}) = \sum_{n\geq i} \alpha_{n,i}\, \mathbf{R}_i^n(\mathbf{t}) \bullet_0 (\boldsymbol{\nabla}_i^n f(\mathbf{r})). \tag{6}
$$

which can also be interpreted as a Taylor series written in spherical derivatives. So, the ordinary differential shift operator $\tau = e^{\mathbf{t}^T \nabla}$ has the spherical analogon

$$
\tau = \sum_{n\geq i} \alpha_{n,i}\, \mathbf{R}_i^n(\mathbf{t}) \bullet_0 \boldsymbol{\nabla}_i^n,
$$

which is much more convenient when we have to deal with rotations. We further can find that

$$
(\boldsymbol{\nabla}_{i'}^{n'} R_{i,m}^n)(\mathbf{0}) = \frac{\delta_{n,n'}\delta_{i,i'}}{\alpha_{n,i}} \mathbf{e}_m^{n-i}
$$

which can be verified by setting $f = R_{i,m}^n$ in equation (6) and comparing the coefficients.

4 Harmonic Subspace

In this section we will consider the subspace $\mathcal{H}$ of harmonic polynomials of $\mathcal{T}_0$. A function $f \in \mathcal{T}_0$ is called harmonic if $\Delta f = 0$. The harmonic functions play an important role in the context of spherical derivatives.

In fact, the space $\mathcal{H}$ together with the inner product introduced below is a reproducing kernel Hilbert space. First, we will derive the reproducing kernel of $\mathcal{H}$ with respect to a Gaussian weighted inner product on $\mathcal{T}_0$. We follow the approach given in [13]. Actually, the reproducing kernel provides an orthogonal projection $\mathcal{H}$. Then, we examine the harmonic projection of a plane wave. This knowledge will be important to understand the harmonic part of the Gabor filter, which is much cheaper to compute as we will later see. Then, we show that the spherical derivatives $\boldsymbol{\nabla}^\ell$ of a Gaussian are just Gaussian-windowed solid harmonics, which are a basis of the harmonic polynomials. This fact will later enable us to compute the Gabors in an efficient manner by a superposition of spherical derivatives. We will also show that the spherical derivative of a Gaussian-smoothed function is equivalent to a convolution with Gaussian-windowed solid harmonics.

4.1 Reproducing Kernel of $\mathcal{H}$

First we have to establish an inner product on $\mathcal{T}_0$ to enable us to speak of orthogonality. We introduce the Gaussian weighted inner product on $\mathcal{T}_0$ by

$$\langle f, g\rangle_\mu = \int_{\mathbb{R}^3} \overline{f(\mathbf{r})} g(\mathbf{r}) e^{-r^2/2} d\mathbf{r} = \int_{\mathbb{R}^3} \overline{f(\mathbf{r})} g(\mathbf{r}) d\mu(\mathbf{r}),$$

where we normalize the standard measure $d\mathbf{r}$ and $d\mu(\mathbf{r})$, respectively, such that $\int_{\mathbb{R}^3} e^{-r^2/2} d\mathbf{r} = \int_{\mathbb{R}^3} d\mu(\mathbf{r}) = 1$. The solid harmonics $\mathbf{R}^n := \mathbf{R}^n_0$ build an orthogonal basis (with respect to $\langle \cdot, \cdot\rangle_\mu$) of the space $\mathcal{H} \subset \mathcal{T}_0$ of harmonic polynomials on $\mathbb{R}^3$. We want to find an expression for the reproducing kernel K of $\mathcal{H}$ and so an orthogonal projection onto $\mathcal{H}$. Therefore, we have to find the squared norms of $\mathbf{R}^n$ with respect to μ. If we denote the $2n+1$ components of $\mathbf{R}^n$ as R^n_m then we can find out that

$$\begin{aligned}
\langle R^n_m, R^{n'}_{m'}\rangle_\mu &= \int_{\mathbb{R}^3} r^{n+n'} Y^n_m(\mathbf{r}) \overline{Y^{n'}_{m'}}(\mathbf{r}) d\mu(\mathbf{r}) \\
&= \frac{\delta_{n,n'}\delta_{m,m'}}{2n+1} \int_0^\infty r^{2(n+1)} e^{-r^2/2} dr \\
&= \delta_{n,n'}\delta_{m,m'} \frac{(2n+1)!!}{(2n+1)}
\end{aligned}$$

The radius integral was computed by partial integration. To obtain an expression for the reproducing kernel K of $\mathcal{H}$ we follow [13]. We just have to write out the point evaluation in terms of solid harmonics. Let $f \in \mathcal{H}$ some harmonic function, then

$$\begin{aligned} f(\mathbf{r}') &= \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{(2n+1)}{(2n+1)!!}\langle R_m^n, f\rangle_\mu R_m^n(\mathbf{r}') \\ &= \int_{\mathbb{R}^3}\underbrace{\sum_{n=0}^{\infty}\frac{\mathbf{R}^n(\mathbf{r}') \bullet_0 \mathbf{R}^n(\mathbf{r})}{(2n-1)!!}}_{K(\mathbf{r}',\mathbf{r})} f(\mathbf{r})\, d\mu(\mathbf{r}) \\ &= \int_{\mathbb{R}^3} K(\mathbf{r}',\mathbf{r}) f(\mathbf{r})\, d\mu(\mathbf{r}) \end{aligned}$$

gives an explicit expression for K. Note, that $(-1)!! = 1$. In fact, K also provides an orthogonal projection onto $\mathcal{H}$, i.e. for any $f \in \mathcal{T}_0$ we have that

$$(P_\mathcal{H} f)(\mathbf{r}') := \int_{\mathbb{R}^3} K(\mathbf{r}',\mathbf{r}) f(\mathbf{r})\, d\mu(\mathbf{r}) \in \mathcal{H},$$

which is clear by the definition of K. The orthogonality of $P_\mathcal{H}$ comes due to the symmetry of K. The kernel is related to the so called Bargmann-Fock space [14] for functions on $\mathbb{C}$. The principle presented here is a certain kind of generalization from $\mathbb{C}$ to $\mathbb{R}^3$. The holomorphic functions known from the classical Bargmann-Fock space are replaced by the harmonic functions on $\mathbb{R}^3$ and the reproducing kernel $e^{z\overline{z'}}$ in the Bargmann-Fock space is replaced by the above introduced kernel K.

After some tedious computations one can find that the harmonic projection of a plane wave $e^{\mathbf{ik}^\top \mathbf{r}}$ is just

$$(P_\mathcal{H} e^{\mathbf{ik}^\top \mathbf{r}})(\mathbf{r}) = K(\mathbf{r}, \mathbf{ik}) e^{-k^2/2} \tag{7}$$

which one may also have guessed when looking at the harmonic part of the full spherical Taylor expansion of the plane wave.

4.2 Spherical Gaussian Derivatives

We have seen in the last section that the Gaussian measure plays an important role. The especialness of the Gaussian in the context of harmonic analysis comes due to one important fact: the Gaussian-windowed solid harmonics have a very special behavior with respect to the Fourier transform. Actually, they are the derivatives $\boldsymbol{\nabla}^\ell$ of the Gaussian. We will show this in the following.

Proposition 4.1. *The Gaussian windowed harmonic of width σ is defined as*

$$\mathbf{V}^\ell_\sigma(\mathbf{r}) := \frac{1}{\sigma^3}\left(\frac{-1}{\sigma^2}\right)^\ell \mathbf{R}^\ell(\mathbf{r})e^{-\frac{r^2}{2\sigma^2}},$$

then

$$\widetilde{\mathbf{V}}^\ell_\sigma(\mathbf{k}) = \langle e^{\mathbf{i}\mathbf{k}^\top\mathbf{r}}, \mathbf{V}^\ell(\mathbf{r})\rangle_{L_2} = (\mathbf{i})^\ell \mathbf{R}^\ell(\mathbf{k})e^{-\frac{(\sigma k)^2}{2}}.$$

is the Fourier transformation of $\mathbf{V}^\ell(\mathbf{r})$.

Proof. We start with the definition of the Fourier transform and plug in the spherical harmonic expansion of the plane wave. Then, we integrate out the angular dependend part:

$$\begin{aligned}\int_{\mathbb{R}^3} \mathbf{V}^\ell_\sigma(\mathbf{r})e^{-\mathbf{i}\mathbf{k}^\top\mathbf{r}}d\mathbf{r} &= \int_{\mathbb{R}^3} \mathbf{V}^\ell_\sigma(\mathbf{r})\sum_n (2n+1)(-\mathbf{i})^n\, j_n(kr)\, \mathbf{Y}^n(\mathbf{r}) \bullet_0 \mathbf{Y}^n(\mathbf{k}) \\ &= \frac{(\mathbf{i})^\ell}{\sigma^{2\ell+3}}\mathbf{Y}^\ell(\mathbf{k})\int_0^\infty j_\ell(kr)e^{-\frac{r^2}{2\sigma^2}}r^{\ell+2}dr\end{aligned}$$

The residual radius dependend part is integrated by using the series expansion of the spherical Bessel function:

$$\begin{aligned}\int_0^\infty j_\ell(kr)e^{-\frac{r^2}{2\sigma^2}}r^{2+\ell}dr &= \sum_{n=0}^\infty \frac{(-1)^n k^{2n+\ell}}{2^n n!(2(n+\ell)+1))!!}\underbrace{\int_0^\infty r^{2n+2\ell+2}e^{-\frac{r^2}{2\sigma^2}}dr}_{\sigma^{2(n+\ell)+3}(2(n+\ell)+1)!!} \\ &= \sum_{n=0}^\infty \frac{(-1)^n k^{2n+\ell}}{2^n n!\sigma^{-2(n+\ell)-3}} = \sigma^{2\ell+3}k^\ell e^{-\frac{(\sigma k)^2}{2}}\end{aligned}$$

So we arrive at

$$\widetilde{\mathbf{V}}^\ell_\sigma(\mathbf{k}) = \int_{\mathbb{R}^3} \mathbf{V}^\ell_\sigma(\mathbf{r})e^{-\mathbf{i}\mathbf{k}^\top\mathbf{r}}d\mathbf{r} = (\mathbf{i}k)^\ell \mathbf{Y}^\ell(\mathbf{k})e^{-\frac{(\sigma k)^2}{2}}$$

which proves the assertion. □

In fact, for $\sigma = 1$ the $\mathbf{V}^\ell$s are eigenfunctions of the Fourier transformation with eigenvalue $(-\mathbf{i})^\ell$. Using the above proposition it is also easy to show that the $\mathbf{V}^\ell$ are just the ℓth order spherical derivatives of a Gaussian.

Corollary 4.2 (Spherical Gaussian Derivative)**.** *The spherical derivative $\boldsymbol{\nabla}^\ell$ of a Gaussian computes to*

$$\boldsymbol{\nabla}^\ell e^{-\frac{r^2}{2\sigma^2}} = \sigma^3 \mathbf{V}^\ell_\sigma(\mathbf{r}) = \left(-\frac{1}{\sigma^2}\right)^\ell \mathbf{R}^\ell(\mathbf{r})\, e^{-\frac{r^2}{2\sigma^2}}$$

This also implies that for small σ the inner products with such $\mathbf{V}^\ell_\sigma$ tend towards the derivative, meaning

$$(-1)^\ell \langle \mathbf{V}^\ell_\sigma, f\rangle_{L_2} \overset{\sigma\to 0}{\longrightarrow} \left(\boldsymbol{\nabla}^\ell f\right)|_{\mathbf{r}=0}$$

for some $f \in \mathcal{T}_0$. Another implication is that convolutions with the $\mathbf{V}^\ell_\sigma$ are derivatives of smoothed functions

Corollary 4.3 (Smooth Derivatives)**.** *Let $f \in \mathcal{T}_0$, then we can show that*

$$\mathbf{V}^\ell_\sigma * f = \boldsymbol{\nabla}^\ell f_s,$$

where $f_s = \frac{1}{\sigma^3} e^{-\frac{r^2}{2\sigma^2}} * f$.

Proof. In the Fourier domain this assertion is easy to realize:

$$\begin{aligned}\widetilde{\mathbf{V}}^\ell_{\sigma_1}\widetilde{f} &= \left((\mathbf{i}k)^\ell \mathbf{Y}^\ell(\mathbf{k}) e^{-\frac{(\sigma k)^2}{2}}\right)\widetilde{f} \\ &= (\mathbf{i}k)^\ell \mathbf{Y}^\ell(\mathbf{k}) \bullet_\ell \underbrace{\left(e^{-\frac{(\sigma k)^2}{2}}\widetilde{f}\right)}_{\tilde{f}_s} = \widetilde{\boldsymbol{\nabla}}^\ell \widetilde{f}_s.\end{aligned}$$

It is just the associativity and commutatitivity of convolutions. ☐

5 Gabor-like Filters

The above introduced theory about spherical derivatives enables us to compute steerable Gabor filters by repeated applications of differentiation in an efficient manner. We will show that an application of an imaginary shift onto an ordinary Gaussian function results in a Gabor like function, i.e. a Gaussian windowed plane wave. Due to the associativity of differentiation and convolution it is possible to perform a convolution with a Gabor equivalently by a convolution with a

Gaussian followed by the imaginary shift, which can be achieved by the proposed differential operator.

To reduce the computational load we further show that harmonic projections of these Gabor filters are much less expensive to compute. We will show that the reproducing kernel of $\mathcal{H}$ coincides with this harmonic projection of the Gabor.

5.1 Complex Arguments

Any analytical function $\mathbb{R}^3 \to \mathbb{R}$ can be easily augmented to complex arguments in $\mathbb{C}^3$ by formally substituting the real variables in the Taylor series with complex ones. In spherical coordinates it is very much the same but not so obvious. Recall, the spherical shift formula (6). The formal replacement which is equivalent to the cartesian case is to assume the following identity

$$\mathbf{R}^n_i(\mathbf{ik}) = (\mathbf{i})^{n+i}\, \mathbf{R}^n_i(\mathbf{k})$$

for any $\mathbf{k} \in \mathbb{R}^3$. One may wonder that we can 'assume' this identity. Realize, that we can construct any $\mathbf{R}^n_i(\mathbf{k})$ by repeated $\bullet$-products with $\mathbf{Sk}$. So, it is no problem to replace $\mathbf{Sk}$ by $\mathbf{S}(\mathbf{ik}) = \mathbf{iSk}$ in a formal manner and obtain the above formula. Actually, be doing so we went out V_1 which is a real vector space. and also $\mathbf{R}^n_i(\mathbf{ik})$ is not element of V_{n-i} anymore.

Note, that the formal substitution is only compliant with our formulas from before when the squared magnitude of the vector $\mathbf{ik}$ is computed by $-k^2 = (\mathbf{S}(\mathbf{ik})) \bullet_0 (\mathbf{S}(\mathbf{ik}))$ or in other words $-k^2 = (\mathbf{ik})^T(\mathbf{ik})$. This means that although we augmented the $\mathbb{R}^3$ to $\mathbb{C}^3$ we still have to use the standard Euclidean inner product $\mathbf{k}^T\mathbf{k}$ rather than the Hermitian $\mathbf{k}^\top\mathbf{k}$. This is essential for the further considerations.

Now, using the above relation we can evaluate any analytic function at complex arguments via the formula:

$$f(\mathbf{r} + \mathbf{ik}) = \sum_{n \geq i} (\mathbf{i})^{n+i} \alpha_{n,i}\, \mathbf{R}^n_i(\mathbf{k}) \bullet_0 \boldsymbol{\nabla}^n_i f(\mathbf{r}).$$

where $\mathbf{k} \in \mathbb{R}^3$. We will now use this formula to generate a Gabor filter.

5.2 Gabor filter

We can use a pure imaginary shift to generate a Gabor function out of a Gaussian. Let us say that $\tau_{\mathbf{ik}}$ shifts by $\mathbf{ik}$ and let $g(\mathbf{r}) := e^{-\mathbf{r}^T\mathbf{r}/2}$ be a Gaussian, then we have that

$$
\begin{aligned}
G_{\mathbf{k}}(\mathbf{r}) &= (\tau_{\mathbf{ik}}g)(\mathbf{r}) = e^{-(\mathbf{r}+\mathbf{ik})^T(\mathbf{r}+\mathbf{ik})/2} \\
&= g(\mathbf{r})\, e^{-\mathbf{ik}^T\mathbf{r}} e^{\mathbf{k}^T\mathbf{k}/2} \\
&= g(\mathbf{r})\, e^{-\mathbf{ik}^T\mathbf{r}}\, g(\mathbf{ik}) \qquad (8)
\end{aligned}
$$

which means that $G_{\mathbf{k}}$ is a spherical Gabor function with frequency $-\mathbf{k}$ centered at the origin. Note, that we actually used the fact that the inner product $\mathbf{k}^T\mathbf{k}$ in the argument of the exponential is the Euclidean one. As a remark, we do not have to bother about $g(\mathbf{ik})$, it is just depending on k.

To compute a convolution of an arbitrary function f with a Gabor $G_{\mathbf{k}}$ we can use that convolutions and differentiation are associative

$$G_{\mathbf{k}} * f = (\tau_{\mathbf{ik}}g) * f = \tau_{\mathbf{ik}}(g * f) = \tau_{\mathbf{ik}} f_s.$$

Thus, a convolution with a Gabor is equivalent to a convolution with an ordinary Gaussian and then performing a complex shift $\tau_{\mathbf{ik}}$. This shift can written as the already proposed differential operator

$$(\tau_{\mathbf{ik}} f_s)(\mathbf{r}) = \sum_{n \geq i} (\mathbf{i})^{n+i} \alpha_{n,i}\, \mathbf{R}_i^n(\mathbf{k}) \bullet_0 \boldsymbol{\nabla}_i^n f_s(\mathbf{r}).$$

Our goal is to compute convolutions with Gabors with fixed frequency magnitude k, while the orientational part should be steerable. From a computational perspective we have to do the following: compute the smooth derivative images $\boldsymbol{\nabla}_i^n f_s$ and collect those with the same tensor rank in one sum

$$\mathbf{A}^\ell = \sum_{n-i=\ell} (-1)^i \alpha_{n,i}\, k^{n+i} \boldsymbol{\nabla}_i^n f_s. \qquad (9)$$

Now, to compute the response of the Gabor filter in a specific direction $\mathbf{n} = \mathbf{k}/k$, or directions that depend on the position $\mathbf{n}(\mathbf{r})$, we just have to compute

$$\sum_\ell (\mathbf{i})^\ell \underbrace{\mathbf{Y}^\ell(\mathbf{n}(\mathbf{r}))}_{\mathbf{E}^\ell(\mathbf{r})} \bullet_0 \mathbf{A}^\ell(\mathbf{r}), \qquad (10)$$

where $\mathbf{E}^\ell(\mathbf{r})$ is a sort of 'direction'-field. The question of how to determine $n(\mathbf{r})$ or $\mathbf{E}^\ell(\mathbf{r})$, respectively, depends on the application. Kovacs et al [15] have shown that it is possible to perform a steering, e.g. with respect to the maximal response, efficiently by a FFT. In our case we need a 2D FFT due to the rotation symmetry of the template.

We already mentioned that the imaginary shift leads out of the real vector spaces V_ℓ. From a computational point of view this is very disadvantageous because we cannot handle V_ℓ as a space of $2\ell+1$ real numbers anymore but have to store $2(2\ell+1)$ numbers which doubles the memory consumption. But in the above formulation the tensor fields $\mathbf{A}^\ell$ and $\mathbf{E}^\ell$ are still elements of V_ℓ. We were able to carry out the complex superposition to the final summation in equation (10), so we can still work with tensors of real dimension $2\ell+1$.

5.3 Harmonic Projection

In order to compute the steerable Gabor responses the computational most expensive part is the computation of the derivatives $\boldsymbol{\nabla}_i^n$ in the sum for $\mathbf{A}^\ell$. It would be grateful if one can restrict the computation to a certain subset, e.g let $i=0$. In fact, just considering $\boldsymbol{\nabla}^n$ is equivalent to projecting onto the subspace of harmonic functions $\mathcal{H}$. Actually, the resulting convolution kernel is just an imaginary evaluation of the harmonic reproducing kernel K.

In other words, the restriction on the coefficients with $i=0$ in equation (9) is equivalent to a convolution of the function f with the kernel

$$H_{\mathbf{k}}(\mathbf{r}) := g(\mathbf{r})K(\mathbf{r}, -\mathbf{ik})$$

Note the similarity of this expression to equation (8). We just exchanged the Gaussian windowed plane wave $e^{-\mathbf{ik}^\top\mathbf{r}}g(\mathbf{ik})$ by its projection on the harmonic subspace, namely $K(\mathbf{r}, -\mathbf{ik})$ (compare to equation (7)). To see the equivalence consider the following computation

$$\begin{aligned}(H_{\mathbf{k}} * f)(\mathbf{r}) &= (g(\mathbf{r})K(\mathbf{r}, -\mathbf{ik})) * f(\mathbf{r}) \\ &= \sum_\ell (\mathbf{i})^\ell \frac{\mathbf{R}^\ell(\mathbf{k}) \bullet_0 (\boldsymbol{\nabla}^\ell f_s)(\mathbf{r})}{(2\ell-1)!!} \\ &= \sum_\ell (\mathbf{i})^\ell\, \mathbf{Y}^\ell(\mathbf{k}) \bullet_0 \mathbf{A}^\ell(\mathbf{r}),\end{aligned}$$

with

$$\mathbf{A}^\ell = \frac{k^\ell}{(2\ell-1)!!}\boldsymbol{\nabla}^\ell f_s \text{ and } f_s = g * f$$

for some fixed width parameter k. To get from the first to the second line we need Corollary 4.3 and the definition of K. Comparing this result to equation (9) and (10) it is just the restriction on the beforementioned coefficients.

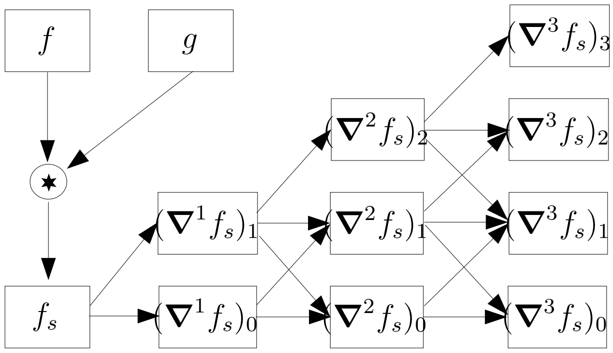

Figure 1: Computation of the spherical derivatives. The arrow indicate the dependencies. For example, to compute $(\nabla^3 f_s)_1$ one have to compute finite differences of $(\nabla^2 f_s)_0, (\nabla^2 f_s)_1, (\nabla^2 f_s)_2$ and make an appropriate superposition of them.

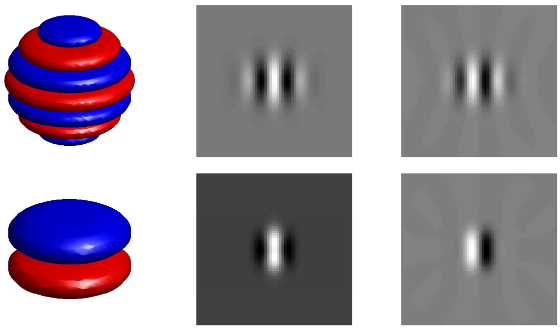

Figure 2: On the top the real part of a Gabor in surface representation (left), a slice through the x-plane of the real part (middle), and imaginary part (right). On the bottom the same for the imaginary part of the harmonic projection of the Gabor. Both were computed with the proposed scheme with an expansion degree up to $\ell_{\text{max}} = 10$ on a 64^3-grid.

6 Implementation

We already explained the workflow of the algorithm roughly. One question remain: How we practically compute the spherical derivatives?

6.1 Spherical Derivatives

For a fast computation of the higher order derivatives we can use finite differences in a repeated manner. In [3] this issue was already discussed for 2D. Alternating forward backward differences seem to be good choice. But central differences are already sufficient for low orders. Of course, the accuracy always depends on the size of the Gaussian smooth which is applied beforehand. In Figure 1 the workflow of the computation of the homogenous derivatives $\boldsymbol{\nabla}^\ell f_s$ is depicted. The round brackets with the subscript indicate the component of the vector, i.e. $(\boldsymbol{\nabla}^\ell f_s)_m = (\boldsymbol{\nabla}^\ell f)^\top \mathbf{e}^\ell_m$. As already mentioned we can restrict the computations on the components with $m \geq 0$ due to the real nature of V_ℓ. The computation of a $(\boldsymbol{\nabla}^{\ell+1} f_s)_m$ only involves derivatives of $(\boldsymbol{\nabla}^\ell f_s)_{m-1}, (\boldsymbol{\nabla}^\ell f_s)_m, (\boldsymbol{\nabla}^\ell f_s)_{m+1}$ due to the selection rules of the Clebsch Gordan coefficients $\langle(\ell+1)m|\ell n, 1(m-n)\rangle$. So, the computation is rather fast and only linear in the number of coefficients.

To compute the derivatives $\boldsymbol{\nabla}^n_i f_s$ in general there are two ways: we can apply the spherical down-derivative in the same manner as the up-derivative by finite differences; or, we can use the Proposition 3.3 and carry out the convolution with multiple Laplacians of Gaussians as a preprocessing step, i.e. we compute $f^i_s = f*(\Delta^i g)$ in an appropriate way and then apply the $\boldsymbol{\nabla}^\ell$ as above and obtain $\boldsymbol{\nabla}^n_i f_s = \boldsymbol{\nabla}^{n-i} f^i_s$. This way is more accurate than the first one but also more expensive due to the explicit convolution with the $\Delta^i g$.

Using the first, more rough approach, we have a complexity of $\mathcal{O}(N \cdot \ell^3_{\text{max}})$ where N is the number of voxels. One have to compute ℓ^2_{max} derivative images, where each derivative has components of order ℓ_{max}, thus the complexity is of order ℓ^3_{max}. Compare this to the direct approach using convolutions. One have to compute ℓ^2_{max} convolutions which is usually accomplished by a FFT. So, we have a complexity of $\mathcal{O}(N\log(N) \cdot \ell^2_{\text{max}})$ for the classical approach. Note, that for the harmonic projection of the Gabor our approach has complexity of order $\mathcal{O}(N \cdot \ell^2_{\text{max}})$ while the classical approach cannot benefit.

In Figure 2 we give two examples: the Gabor and its harmonic projection. For the computation of the derivatives we used central differences and the more rough approximation of the down-derivative by finite differences. An example concerning the running times: computing the full steerable expansion of a Gabor with $\ell_{\max} = 5$ on a 256^3 grid takes on an *Intel Xeon X5365 / 3Ghz* (4MB Cache, single threaded) about $25s$, where everything was implemented in *C++* without any optimizations like SSE. Using the direct convolution approach with a SSE optimized *FFTW* ("patient") this takes about $30s$.

6.2 Fast Steering

Finally, we address the task of computing the sum $c(\mathbf{n}) = \sum_\ell (\mathbf{i})^\ell \mathbf{A}^\ell \bullet_0 \mathbf{Y}^\ell(\mathbf{n})$ for different $\mathbf{n} \in S^2$ in an efficient way. Note, that we leave out the position dependency comparing to equation (10). Practically we have to apply the following procedure to each voxel (or an appropriate chosen subset). Actually, we can use a 2D FFT to accomplish the task. By parameterizing $\mathbf{r}$ by angles $\theta \in [0, \pi)$ and $\phi \in [0, 2\pi)$ we can write a spherical harmonic by $Y_n^\ell(\theta, \phi) = \sum_{h=-\ell}^{\ell} \mathbf{i}^{n+2h} d_{nh}^\ell d_{h0}^\ell e^{\mathbf{i}(n\phi + h\theta)}$, where d_{nh}^ℓ is the Wigner D-matrix corresponding to a rotation around y-axis by 90 degrees. This decomposition is due to Kovacs et al [15]. Using this, it is easy to find an expression for $c(\theta, \phi)$ of the form $c(\theta, \phi) = \sum_{h,n} c_{nh} e^{\mathbf{i}(n\phi + h\theta)}$ that allows the use of a 2D FFT for a fast computation.

7 Conclusion

In this work we proposed the theoretical foundations of spherical differential calculus which allows us to compute derivatives that are compliant with rotation behavior of the usual spherical harmonic representation. We derived analytical formulas for the Gabor filter and a certain harmonic projection of the Gabor in terms of spherical derivatives. They enable us to compute steerable Gabors in an efficient way.

A Spherical Harmonics

We always use Racah-normalized spherical harmonics. In terms of Legendre polynomials they are written as

$$Y_m^\ell(\phi, \theta) = \sqrt{\frac{(l-m)!}{(l+m)!}} P_m^\ell(\cos(\theta)) e^{\mathbf{i}\phi}$$

Mostly we write $\mathbf{r}/r \in S^2$ instead of (ϕ, θ). The Racah-normalized solid harmonics can be written as

$$R^\ell_m(\mathbf{r}) = \sqrt{(\ell+m)!(\ell-m)!}\sum_{i,j,k}\frac{\delta_{i+j+k,\ell}\delta_{i-j,m}}{i!j!k!2^i2^j}(x-\mathbf{i}y)^j(-x-\mathbf{i}y)^iz^k,$$

where $\mathbf{r} = (x, y, z)$. They are related to spherical harmonics by $R^\ell_m(\mathbf{r})/r^\ell = Y^\ell_m(\mathbf{r}/r)$

B Clebsch Gordan Coefficients

Orthogonality

$$\sum_{j,m}\langle jm|j_1m_1, j_2m_2\rangle\langle jm|j_1m_1', j_2m_2'\rangle = \delta_{m_1,m_1'}\delta_{m_2,m_2'} \tag{11}$$

$$\sum_{m=m_1+m_2}\langle jm|j_1m_1, j_2m_2\rangle\langle j'm'|j_1m_1, j_2m_2\rangle = \delta_{j,j'}\delta_{m,m'} \tag{12}$$

$$\sum_{m_1,m}\langle jm|j_1m_1, j_2m_2\rangle\langle jm|j_1m_1, j_2'm_2'\rangle = \frac{2j+1}{2j_2'+1}\delta_{j_2,j_2'}\delta_{m_2,m_2'} \tag{13}$$

Special Values

$$\langle \ell m|(\ell-\lambda)(m-\mu), \lambda\mu\rangle = \begin{pmatrix} \ell+m \\ \lambda+\mu \end{pmatrix}^{1/2}\begin{pmatrix} \ell-m \\ \lambda-\mu \end{pmatrix}^{1/2}\begin{pmatrix} 2\ell \\ 2\lambda \end{pmatrix}^{-1/2} \tag{14}$$

$$\langle \ell m|(\ell+\lambda)(m-\mu), \lambda\mu\rangle = (-1)^{\lambda+\mu}\begin{pmatrix} \ell+\lambda-m+\mu \\ \lambda+\mu \end{pmatrix}^{1/2} \begin{pmatrix} \ell+\lambda+m-\mu \\ \lambda-\mu \end{pmatrix}^{1/2}\begin{pmatrix} 2\ell+2\lambda+1 \\ 2\lambda \end{pmatrix}^{-1/2} \tag{15}$$

Symmetry

$$\langle jm|j_1m_1, j_2m_2\rangle = \langle j_1m_1, j_2m_2|jm\rangle \tag{16}$$

$$\langle jm|j_1m_1, j_2m_2\rangle = (-1)^{j+j_1+j_2}\langle jm|j_2m_2, j_1m_1\rangle \tag{17}$$

$$\langle jm|j_1m_1, j_2m_2\rangle = (-1)^{j+j_1+j_2}\langle j(-m)|j_1(-m_1), j_2(-m_2)\rangle \tag{18}$$

C Wigner D-Matrix

The components of $\mathbf{D}^\ell_g$ are written D^ℓ_{mn}. They are called the Wigner D-matrix. In Euler angles ϕ, θ, ψ in ZYZ-convention we have

$$D^\ell_{mn}(\phi, \theta, \psi) = e^{\mathbf{i}m\phi} d^\ell_{mn}(\theta) e^{\mathbf{i}n\psi},$$

where $d^\ell_{mn}(\theta)$ are the Wigner d-matrix which is real-valued. Relation to the Clebsch Gordan coefficients:

$$D^\ell_{mn} = \sum_{\substack{m_1+m_2=m \\ n_1+n_2=n}} D^{\ell_1}_{m_1n_1} D^{\ell_2}_{m_2n_2} \langle lm|l_1m_1, l_2m_2\rangle \langle ln|l_1n_1, l_2n_2\rangle \tag{19}$$

$$D^{\ell_1}_{m_1n_1} D^{\ell_2}_{m_2n_2} = \sum_{l,m,n} D^\ell_{mn} \langle lm|l_1m_1, l_2m_2\rangle \langle ln|l_1n_1, l_2n_2\rangle \tag{20}$$

D Spherical Bessel Functions

As Taylor series with $\alpha \in \mathbb{R}$,

$$J_\alpha(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\alpha+1)} \left(\frac{x}{2}\right)^{2n+\alpha}$$

The spherical Bessel function with $m \in \mathbb{N}$ is given by

$$j_m(x) = \sqrt{\frac{\pi}{2x}} J_{m+1/2}(x)$$

as series expansion

$$j_m(x) = \sqrt{\frac{\pi}{4}} \left(\frac{x}{2}\right)^m \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+m+3/2)} \left(\frac{x}{2}\right)^{2n}. \tag{21}$$

or

$$j_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!(2(n+m)+1)!!} x^{2n+m}. \tag{22}$$

D.1 Double Factorial

$$(2\ell+1)!! = \Gamma(\ell+3/2)\frac{2^l}{\sqrt{\pi/4}} = (2\ell+1)(2\ell-1)(2\ell-3)\ldots$$

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