Spherical Tensor Calculus for Local Adaptive Filtering

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1 Introduction

In 3D image processing tensors play an important role. While rank-1 and rank-2 tensors are well understood and commonly used, higher rank tensors are rare. This is probably due to their cumbersome rotation behavior which prevents a computationally efficient use. In this chapter we want to introduce the notion of a spherical tensor which is based on the irreducible representations of the 3D rotation group. In fact, any ordinary cartesian tensor can be decomposed into a sum of spherical tensors, while each spherical tensor has a quite simple rotation behavior. We introduce so called tensorial harmonics that provide an orthogonal basis for spherical tensor fields of any rank. It is just a generalization of the well known spherical harmonics. Additionally we propose a spherical derivative which connects spherical tensor fields of different degree by differentiation.

We will use the proposed theory for local adaptive filtering. By local adaptive filtering we mean that during the filtering process the filter kernels may change their shape and orientation depending on other quantities which were derived from the image. Typically there are two ways to do this which are in a certain sense dual to each other. Consider the classical linear filtering process. There are two interpretation, on the one hand the convolution: each pixel (impulse) in the image is replaced by a predefined filter kernel (impulse response) while the filter kernel itself is weighted by the intensity of the observed pixel. The contribution from all pixels are combined by summation. This is the interpretation we know from signal processing, where the filter kernel is known as the impulse response. For Gaussian filter kernels the physical interpretation of this is simple isotropic diffusion. The second interpretation is to compute a kind of correlation or blurring of the image: at each

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pixel we compute an inner product of the filter kernel with its local neighborhood, i.e. a kind of correlation. If the filter kernel is positive, then it may be interpreted as an average of the surrounding pixels while the filter kernel determines the shape and size of local window in which the average is taken. In the linear case both interpretation are identical up to a point reflection of the filter kernel. But, if the filter kernel is spatially dependend (or local adaptive) both approaches are not identical anymore. Let us formalize this. Let $m(\mathbf{r})$ be the intensity of an image at position $\mathbf{r}$ and $V^{\mathbf{n}}(\mathbf{r})$ a filter kernel at position $\mathbf{r}$, where the superscript $\mathbf{n}$ is a parameter that determines the orientation and shape of the kernel. Now suppose that we have also given a parameter field $\mathbf{n}(\mathbf{r})$, i.e. the appearance of the kernel is spatially dependend. Then, the 'convolution' integral looks as

$$\mathbf{U}_{\text{conv}}(\mathbf{r}) = \int_{\mathbb{R}^3} V^{\mathbf{n}(\mathbf{r}')}(\mathbf{r}-\mathbf{r}')\; m(\mathbf{r}')\; d\mathbf{r}'.$$

It formulates the above described intuition. We attach to each position $\mathbf{r}' \in \mathbb{R}^3$ the filter kernel while the filter kernel depends on the kernel parameter $\mathbf{n}$ at position $\mathbf{r}'$. Then, the filter kernel is weighted by the observed image intensity $m(\mathbf{r}')$ and the contributions from all positions $\mathbf{r}'$ are superimposed additively by the integral. On the other hand we can write down the 'correlation' integral as

$$\mathbf{U}_{\text{corr}}(\mathbf{r}) = \int_{\mathbb{R}^3} V^{\mathbf{n}(\mathbf{r})}(\mathbf{r}'-\mathbf{r})\; m(\mathbf{r}')\; d\mathbf{r}',$$

which again covers the above presented picture. The value of the result at position $\mathbf{r}$ is just the standard innerproduct of the image with filter kernel modified by the parameter $\mathbf{n}(\mathbf{r})$.

The 'convolution'-approach is related to the so called Tensor Voting framework (TV) [5, 7]. In TV the filter kernel is denoted as the voting function and is typically tensor-valued. For example, rank 2 tensors are use to enhance feature images for fiber detection. In TV the intensity image $m(\mathbf{r})$ is interpreted as a probability for the presence of a fiber, while the kernel parameter $\mathbf{n}(\mathbf{r})$ is the orientation of the fiber at the specific position. On the other hand, the 'correlation'-approach is related to anisotropic smoothing filters, which are typically used to denoise images while preserving edges and discontinuities. Here the filter kernel is for example a squeezed Gaussian, tablet like function, which is during the filter process oriented along the intensity gradients. In this way the smoothing is not performed across edges and, hence, the discontinuities are preserved.

In this Chapter we propose how to use spherical tensor calculus to expand the filter kernel in an advantageous manner, such that the orientational steering of the filter kernel can be performed efficiently. For scalar filter kernels this expansion is the well-known Spherical Harmonics expansion. To generalize this idea to tensor-valued images we propose the so called tensorial harmonics. In this way arbitrary filter kernels can be expanded in tensorial harmonics and the computation of filter integral turns out to be a sum of convolutions. Although the convolutions can be computed efficiently by the Fast Fourier Transform, the convolution is still the bottleneck in the computation for very large volumes. Another problem of this approach is the severe memory consumption, because one has to store the tensorial harmonic decomposition in a quite wasteful manner to allow an efficient computation. Hence, we introduce so called spherical derivatives that allow to compute the convolutions with special type of kernels efficiently.

1.1 Related Work

The Tensor Voting (TV) framework was originally proposed by Medioni et al. [5] and has found several applications in low-level vision in 2D and 3D. For example, it is used for perceptual grouping and extraction of line, curves and surfaces [7]. The key idea of TV is to make unreliable measurements more robust by incorporating neighborhood information in a consistent and coherent manner. To compute the TV-integral in reasonable time the initial measurements in TV are typically sparse. Recently, Franken et al. [2] proposed an efficient way to compute a dense Tensor Voting in 2D. The idea makes use of a steerable expansion of the voting field. Steerable filters are an efficient architecture to synthesize filters for arbitrary angles from linear combinations of basis filters [3]. Perona generalized this concept in [8] and introduced a methodology to decompose a given filter kernel optimally in a set of steerable basis filters. The idea of Franken et al. [2] is to use the steerable decomposition of the voting field to compute the voting process by convolutions in an efficient way. Complex calculus and 2D harmonic analysis are the major mathematical tools that make this approach possible.

Anisotropic filtering is a low-level image processing technique that is used to denoise and enhance images. The applied algorithms can be separated into iterative and non-iterative methods. Iterative algorithms [10] are based on solutions of partial differential equations. The motivation of the idea is founded in the physical modelling of an anisotropic diffusion process. The equations are tailored such that particles tend to diffuse along edges rather than across edges. Consequently, the discontinuities of the images are preserved while the isotropic regions are smoothed. The second class of algorithms [13, 4] treats the problem as a local adaptive blurring process. Depending on a local orientation analysis the blurring kernels are steered for each pixels such that the blurring is not performed across edges. In [4] a technique for fast anisotropic filtering in 2D is proposed, unfortunately the idea is not extendable to 3D.

2 Spherical Tensor Analysis

We will assume that the reader is familiar with the basic notions of the harmonic analysis of $SO(3)$. For introductory reading we recommend mostly literature [12, 9] concerning the quantum theory of the angular momentum, while our representation tries to avoid terms from quantum theory to also give the non-physicists a chance for following. See e.g. [6, 11] for introduction from an engineering or mathematical point of view.

In the following we just repeat the basic notions and introduce our notations.

2.1 Preliminaries

Let $\mathbf{D}_g^j$ be the unitary irreducible representation of a $g \in SO(3)$ of order j with $j \in \mathbb{N}$. They are also known as the *Wigner D-matrices* (see e.g. [9]). The representation $\mathbf{D}_g^j$ acts on a vector space V_j which is represented by $\mathbb{C}^{2j+1}$. We write the elements of V_j in bold face, e.g. $\mathbf{u} \in V_j$ and write the $2j+1$ components in unbold face $u_m \in \mathbb{C}$ where $m = -j, \dots j$. For the transposition of a vector/matrix we write $\mathbf{u}^T$; the joint complex conjugation and transposition is denoted by $\mathbf{u}^\top = \overline{\mathbf{u}}^T$. In this terms the unitarity of $\mathbf{D}_g^j$ is expressed by the formula $(\mathbf{D}_g^j)^\top \mathbf{D}_g^j = \mathbf{I}$.

Note, that we treat the space V_j as a real vector space of dimensions $2j+1$, although the components of $\mathbf{u}$ might be complex. This means that the space V_j is only closed under weighted superpositions with real numbers. As a consequence of this we always have that the components are interrelated by $\overline{u_m} = (-1)^m u_{-m}$. From a computational point of view this is an important issue. Although the vectors are elements of $\mathbb{C}^{2j+1}$ we just have to store just $2j+1$ real numbers.

We denote the standard basis of $\mathbb{C}^{2j+1}$ by $\mathbf{e}_m^j$, where the nth component of $\mathbf{e}_m^j$ is δ_{mn}. In contrast, the standard basis of V_j is written as $\mathbf{c}_m^j = \frac{1+\mathbf{i}}{2}\mathbf{e}_m^j + (-1)^m \frac{1-\mathbf{i}}{2}\mathbf{e}_{-m}^j$. We denote the corresponding 'imaginary' space by $\mathbf{i}V_j$, i.e. elements of $\mathbf{i}V_j$ can be written as $\mathbf{iv}$ where $\mathbf{v} \in V_j$. So, elements $\mathbf{w} \in \mathbf{i}V_j$ fulfill $\overline{w_m} = (-1)^{m+1} w_{-m}$. Hence, we can write the space $\mathbb{C}^{2j+1}$ as the direct sum of the two spaces $\mathbb{C}^{2j+1} = V_j \oplus \mathbf{i}V_j$. The standard coordinate vector $\mathbf{r} = (x,y,z)^T \in \mathbb{R}^3$ has a natural relation to elements $\mathbf{u} \in V_1$ by

$$\mathbf{u} = \frac{x-y}{\sqrt{2}}\mathbf{c}_1^1 + z\mathbf{c}_0^1 - \frac{x+y}{\sqrt{2}}\mathbf{c}_{-1}^1 = \begin{pmatrix} \frac{1}{\sqrt{2}}(x-\mathbf{i}y) \\ z \\ -\frac{1}{\sqrt{2}}(x+\mathbf{i}y) \end{pmatrix} = \mathbf{Sr} \in V_1$$

Note, that $\mathbf{S}$ is an unitary coordinate transformation. The representation $\mathbf{D}_g^1$ is directly related to the real valued rotation matrix $\mathbf{U}_g \in SO(3) \subset \mathbb{R}^{3\times 3}$ by $\mathbf{D}_g^1 = \mathbf{S}\mathbf{U}_g\mathbf{S}^\top$.

Definition 2.1 *A function $\mathbf{f} : \mathbb{R}^3 \mapsto \mathbb{C}^{2j+1}$ is called a spherical tensor field of rank j if it transforms with respect to rotations as*

$$(g\mathbf{f})(\mathbf{r}) := \mathbf{D}_g^j \mathbf{f}(\mathbf{U}_g^T \mathbf{r})$$

for all $g \in SO(3)$. The space of all spherical tensor fields of rank j is denoted by $\mathscr{T}_j$.

2.2 Spherical Tensor Coupling

Now, we define a family of bilinear forms that connect tensors of different ranks.

Definition 2.2 *For every* $j \geq 0$ *we define a family of bilinear forms of type*

$$\circ_j : V_{j_1} \times V_{j_2} \mapsto \mathbb{C}^{2j+1}$$

where $j_1, j_2 \in \mathbb{N}$ *has to be chosen according to the triangle inequality* $|j_1 - j_2| \leq j \leq j_1 + j_2$*. It is defined by*

$$(\mathbf{e}_m^j)^\top (\mathbf{v} \circ_j \mathbf{w}) := \sum_{m=m_1+m_2} \langle jm \mid j_1 m_1, j_2 m_2 \rangle v_{m_1} w_{m_2}$$

where $\langle jm \mid j_1 m_1, j_2 m_2 \rangle$ *are the Clebsch-Gordan coefficients.*

For references concerning the Clebsch-Gordan coefficients see in the appendix. The characterizing property of these products is that they respect the rotations of the arguments, namely

Proposition 2.3 *Let* $\mathbf{v} \in V_{j_1}$ *and* $\mathbf{w} \in V_{j_2}$*, then for any* $g \in SO(3)$

$$(\mathbf{D}_g^{j_1} \mathbf{v}) \circ_j (\mathbf{D}_g^{j_2} \mathbf{w}) = \mathbf{D}_g^j (\mathbf{v} \circ_j \mathbf{w})$$

holds.

Proof. The components of the left-hand side look as

$$\begin{aligned} &(\mathbf{e}_m^j)^\top ((\mathbf{D}_g^{j_1} \mathbf{v}) \circ_j (\mathbf{D}_g^{j_2} \mathbf{w})) \\ &\quad = \sum_{\substack{m=m_1+m_2 \\ m_1' m_2'}} \langle jm | j_1 m_1, j_2 m_2 \rangle D^{j_1}_{m_1 m_1'} D^{j_2}_{m_2 m_2'} v_{m_1'} w_{m_2'} \end{aligned}$$

First, one have to insert the identity by using orthogonality relation (17) with respect to m_1' and m_2'. Then we can use relation (25) and the definition of $\circ_j$ to prove the assertion.

Proposition 2.4 *If* $j_1 + j_2 + j$ *is even, than* $\circ$ *is symmetric, otherwise antisymmetric. The spaces* V_j *are closed for the symmetric product, for the antisymmetric product this is not the case.*

$$\begin{aligned} j + j_1 + j_2 \text{ is even } &\Rightarrow \mathbf{v} \circ_j \mathbf{w} \in V_j \\ j + j_1 + j_2 \text{ is odd } &\Rightarrow \mathbf{v} \circ_j \mathbf{w} \in \mathbf{i} V_j, \end{aligned}$$

where $\mathbf{v} \in V_{j_1}$ *and* $\mathbf{w} \in V_{j_2}$*.*

Proof. The symmetry and antisymmetry is founded in the symmetry properties of the Clebsch-Gordan coefficients in equation (23). To show the closure property consider

$$\begin{aligned}
(\mathbf{e}^j_m)^\top \overline{\mathbf{v} \circ_j \mathbf{w}} &= \sum_{m=m_1+m_2} \langle jm|j_1m_1, j_2m_2\rangle \overline{v_{m_1} w_{m_2}} \\
&= \sum_{m=m_1+m_2} (-1)^m \langle jm|j_1m_1, j_2m_2\rangle v_{-m_1} w_{-m_2} \\
&= \sum_{m=m_1+m_2} (-1)^{m+j+j_1+j_2} \langle j(-m)|j_1m_1, j_2m_2\rangle v_{m_1} w_{m_2} \\
&= (-1)^{m+j+j_1+j_2} (\mathbf{e}^j_{-m})^\top \overline{\mathbf{v} \circ_j \mathbf{w}},
\end{aligned}$$

where we used the symmetry property given in equation (24). Hence, we have for even $j + j_1 + j_2$ the 'realness' condition complying to V_j and for odd $j + j_1 + j_2$ the 'imaginaryness' condition for $\mathbf{i}V_j$, which prove the statements.

We will later see that the symmetric product plays an important role, in particular, because we can normalize it in an special way such that it shows a more gentle behavior with respect to the spherical harmonics.

Definition 2.5 *For every $j \geq 0$ with $|j_1 - j_2| \leq j \leq j_1 + j_2$ and even $j + j_1 + j_2$ we define a family of symmetric bilinear forms by*

$$\mathbf{v} \bullet_j \mathbf{w} := \frac{1}{\langle j0|j_10, j_20\rangle} \mathbf{v} \circ_j \mathbf{w}$$

For the special case $j = 0$ the arguments have to be of the same rank due to the triangle inequality. Actually in this case the symmetric product coincides with the standard inner product

$$\mathbf{v} \bullet_0 \mathbf{w} = \sum_{m=-j}^{m=j} (-1)^m v_m w_{-m} = \mathbf{w}^\top \mathbf{v},$$

where j is the rank of $\mathbf{v}$ and $\mathbf{w}$.

Proposition 2.6 *The products $\circ$ and $\bullet$ are associative in the following manner.*

$$\mathbf{v}^{j_1} \circ_\ell (\mathbf{w}^{j_2} \circ_{j_2+j_3} \mathbf{y}^{j_3}) = (\mathbf{v}^{j_1} \circ_{j_1+j_2} \mathbf{w}^{j_2}) \circ_\ell \mathbf{y}^{j_3} \tag{1}$$

holds if $j_1 + j_2 + j_3 = \ell$. And

$$\mathbf{v}^{j_2} \circ_\ell (\mathbf{w}^{j_1} \circ_{j_1+j_3} \mathbf{y}^{j_3}) = (\mathbf{v}^{j_1} \circ_{j_2-j_1} \mathbf{w}^{j_2}) \circ_\ell \mathbf{y}^{j_3} \tag{2}$$

holds with $\ell = j_2 - (j_1 + j_3) \geq 0$.

Proof. Both statements are proved by using the explicit formulas for the special cases of the Clebsch-Gordan coefficients as given in equation (20) and (21).

The introduced product can also be used to combine tensor fields of different rank by point-wise multiplication.

Proposition 2.7 *Let $\mathbf{v} \in \mathscr{T}_{j_1}$ and $\mathbf{w} \in \mathscr{T}_{j_2}$ and j chosen such that $|j_1 - j_2| \leq j \leq j_1 + j_2$, then*

$$\mathbf{f}(\mathbf{r}) = \mathbf{v}(\mathbf{r}) \circ_j \mathbf{w}(\mathbf{r})$$

is in $\mathscr{T}_j$*, i.e. a tensor field of rank* j*.*

In fact, there is another way to combine two tensor fields: by convolution. The evolving product respects the translation in a different sense.

Proposition 2.8 *Let* $\mathbf{v} \in \mathscr{T}_{j_1}$ *and* $\mathbf{w} \in \mathscr{T}_{j_2}$ *and* j *chosen such that* $|j_1 - j_2| \leq j \leq j_1 + j_2$*, then*

$$(\mathbf{v} \tilde{\circ}_j \mathbf{w})(\mathbf{r}) := \int_{\mathbb{R}^3} \mathbf{v}(\mathbf{r}' - \mathbf{r}) \circ_j \mathbf{w}(\mathbf{r}')\, d\mathbf{r}'$$

is in $\mathscr{T}_j$*, i.e. a tensor field of rank* j*.*

2.3 Relation to Cartesian Tensors

The correspondence of spherical and cartesian tensors of rank 0 is trivial. For rank 1 it is just the matrix $\mathbf{S}$ that connects the real-valued vector $\mathbf{r} \in \mathbb{R}^3$ with the spherical coordinate vector $\mathbf{u} = \mathbf{S}\mathbf{r} \in V_1$. For rank 2 the consideration gets more intricate. Consider a real-valued cartesian rank-2 tensor $\mathbf{T} \in \mathbb{R}^{3\times 3}$ and the following unique decomposition

$$\mathbf{T} = \alpha \mathbf{I}_3 + \mathbf{T}_{\text{anti}} + \mathbf{T}_{\text{sym}},$$

where $\alpha \in \mathbb{R}$, $\mathbf{T}_{\text{anti}}$ is an antisymmetric matrix and $\mathbf{T}_{\text{sym}}$ a traceless symmetric matrix. In fact, this decomposition follows the same manner as the spherical tensor decomposition. A rank 0 spherical tensor corresponds to the identity matrix in cartesian notation, while the rank 1 spherical tensor to a antisymmetric 3×3 matrix or, equivalently, to a vector. The rank 2 spherical tensor corresponds to a traceless, symmetric matrix. Let us consider the spherical decomposition. For convenience let $\mathbf{T}^s = \mathbf{S}\mathbf{T}\mathbf{S}^\top$, then the components of the corresponding spherical tensors $\mathbf{b}^j \in V_j$ with $j = 0, 1, 2$ look as

$$b^j_m = \sum_{m_1+m_2=m} \langle 1m_1, 1m_2 | jm \rangle (-1)^{m_1} T^s_{(-m_1)m_2},$$

where $\mathbf{b}^0$ corresponds to α, $\mathbf{b}^1$ to $\mathbf{T}_{\text{anti}}$ and $\mathbf{b}^2$ to $\mathbf{T}_{\text{sym}}$. The inverse of this 'cartesian to spherical'-transformation is

$$T^s_{m_1 m_2} = \sum_{j=0}^{2} \sum_{m=-j}^{m=j} \langle 1(-m_1), 1m_2 | jm \rangle (-1)^{m_1} b^j_m.$$

In particular, consider a cartesian symmetric 2-tensor and its eigensystem. In spherical tensor notation the spherical tensor $\mathbf{b}^2$ is decomposed into products of three 1-tensors $\mathbf{v}_k \in V_1$ as

$$\mathbf{b}^2 = \sum_{k=-1}^{1} \lambda_k\, \mathbf{v}_k \circ_2 \mathbf{v}_k,$$

where $\mathbf{v}_k$ are the eigenvectors of $\mathbf{T}^s$ and λ_k the eigenvalues. Note that $\mathbf{b}^2$ is invariant against a common shift of the eigenvalues by some offset γ. It is 'traceless' in sense that

$$\sum_{k=-1}^{1} \mathbf{v}_k \circ_2 \mathbf{v}_k = \mathbf{0},$$

for any set of orthogonal vectors $\mathbf{v}_{-1}, \mathbf{v}_0, \mathbf{v}_1 \in V_1$. This offset, namely the trace of $\mathbf{T}$ is covered by the zero-rank $\mathbf{b}^0$. It corresponds to the 'ballness' or 'isotropy' of $\mathbf{T}$.

2.4 Spherical Harmonics

We denote the well-known spherical harmonics by $\mathbf{Y}^j : S^2 \to V_j$ (see appendix). We always, write $\mathbf{Y}^j(\mathbf{r})$, where $\mathbf{r}$ may be an element of $\mathbb{R}^3$, but $\mathbf{Y}^j(\mathbf{r})$ is independent of the magnitude of $r = \|\mathbf{r}\|$, i.e. $\mathbf{Y}^j(\lambda \mathbf{r}) = \mathbf{Y}^j(\mathbf{r})$ for any $\lambda \in \mathbb{R}$. We know that the $\mathbf{Y}^j$ provide an orthogonal basis of scalar function on the 2-sphere S^2. Thus, any real scalar field $f \in \mathscr{T}_0$ can be expanded in terms of spherical harmonics in an unique manner:

$$f(\mathbf{r}) = \sum_{j=0}^{\infty} \mathbf{a}^j(r)^\top \mathbf{Y}^j(\mathbf{r}),$$

where the $\mathbf{a}^j(r)$ are expansion coefficients just depending on the radius $r = \|\mathbf{r}\|$. In the following, we always use Racah's normalization (also known as semi-Schmidt normalization), i.e.

$$\langle Y^j_m, Y^{j'}_{m'} \rangle = \frac{1}{4\pi} \int_{S^2} Y^j_m(\mathbf{s})\, \overline{Y^{j'}_{m'}}(\mathbf{s}) d\mathbf{s} = \frac{1}{2j+1} \delta_{jj'} \delta_{mm'}$$

where the integral ranges over the 2-sphere using the standard measure. One important property of the Racah-normalized spherical harmonics is that $\mathbf{Y}^{j\top} \mathbf{Y}^j = 1$. Another important and useful property is that

$$\mathbf{Y}^j = \mathbf{Y}^{j_1} \bullet_j \mathbf{Y}^{j_2} \tag{3}$$

if $j + j_1 + j_2$ is even. We can use this formula to iteratively compute higher order $\mathbf{Y}^j$ from given lower order ones. Note that $\mathbf{Y}^0 = 1$ and $\mathbf{Y}^1 = \mathbf{S}\mathbf{r}$, where $\mathbf{r} \in S^2$.

The spherical harmonics have a variety of nice properties. One of the most important ones is that each $\mathbf{Y}^j$, interpreted as a tensor field of rank j is a fix-point with respect to rotations, i.e.

$$(g\mathbf{Y}^j)(\mathbf{r}) = \mathbf{D}^j_g \mathbf{Y}^j(\mathbf{U}_g^T \mathbf{r}) = \mathbf{Y}^j(\mathbf{r})$$

or in other words $\mathbf{Y}^j(\mathbf{U}_g \mathbf{r}) = \mathbf{D}^j_g \mathbf{Y}^j(\mathbf{r})$. A consequence of this is that the expansion coefficients of the rotated function $(gf)(\mathbf{r}) = f(\mathbf{U}_g^T \mathbf{r})$ just look as $\mathbf{D}^j_g \mathbf{a}^j(r)$.

Note that the spherical harmonics arise as solutions of the Laplace equation $\Delta f = 0$. One set of solutions are the homogeneous polynomials

$$\mathbf{R}^j(\mathbf{r}) := r^j \mathbf{Y}^j(\mathbf{r}),$$

i.e. the $\mathbf{R}^j$ fulfill $\mathbf{R}^j(\lambda \mathbf{r}) = \lambda^j \mathbf{R}^j(\mathbf{r})$ and the components solve the Laplace equation $\Delta R^j_m = 0$. In literature these functions are called the solid harmonics. They will get important in the context of the spherical tensor derivatives.

3 Tensorial Harmonic Expansion

We propose to expand a tensor field $\mathbf{f} \in \mathscr{T}_\ell$ of rank ℓ as follows

$$\mathbf{f}(\mathbf{r}) = \sum_{j=0}^{\infty} \sum_{k=-\ell}^{k=\ell} \mathbf{a}^j_k(r) \circ_\ell \mathbf{Y}^j(\mathbf{r}),$$

where $\mathbf{a}^j_k(r) \in \mathscr{T}_{j+k}$ are expansion coefficients. Note, that for $\ell = 0$ the expansion coincides with the ordinary scalar expansion from above. We can further observe that

$$\begin{aligned}(g\mathbf{f})(\mathbf{r}) &= \mathbf{D}^\ell_g \mathbf{f}(\mathbf{U}_g^\top \mathbf{r}) \\ &= \sum_{j=0}^{\infty} \sum_{k=-\ell}^{k=\ell} (\mathbf{D}^{j+k}_g \mathbf{a}^j_k(r)) \circ_\ell \mathbf{Y}^j(\mathbf{r}) \end{aligned} \tag{4}$$

i.e. a rotation of the tensor field affects the expansion coefficients $\mathbf{a}^j_k$ to be transformed by $\mathbf{D}^{j+k}_g$.

By setting $\mathbf{a}^j_k(r) = \sum_{m=-(j+k)}^{m=j+k} a^j_{km}(r) \mathbf{e}^{j+k}_m$ we can identify the functional basis $\mathbf{Z}^j_{km}$ as

$$\mathbf{f}(\mathbf{r}) = \sum_{j=0}^{\infty} \sum_{k=-\ell}^{k=\ell} \sum_{m=-(j+k)}^{m=j+k} a^j_{km}(r) \underbrace{\mathbf{e}^{j+k}_m \circ_\ell \mathbf{Y}^j(\mathbf{r})}_{\mathbf{Z}^j_{km}},$$

Proposition 3.1 (Tensorial Harmonics) *The functions $\mathbf{Z}^j_{km} : S^2 \mapsto V_\ell$ provide an complete and orthogonal basis of the angular part of $\mathscr{T}_\ell$, i.e.*

$$\int_{S^2} (\mathbf{Z}^j_{km}(\mathbf{s}))^\top \mathbf{Z}^{j'}_{k'm'}(\mathbf{s}) d\mathbf{s} = \frac{4\pi}{N_{j,k}} \delta_{j,j'} \delta_{k,k'} \delta_{m,m'},$$

where

$$N_{j,k} = \frac{1}{2\ell+1}(2j+1)(2(j+k)+1).$$

The functions $\mathbf{Z}^j_{km}$ are called the tensorial harmonics.

Proof. We first show the orthogonality by elementary calculations:

$$\begin{aligned}
&\frac{1}{4\pi}\int_{S^2}(\mathbf{Z}^j_{km}(\mathbf{s}))^\top \mathbf{Z}^{j'}_{k'm'}(\mathbf{s})d\mathbf{s}\\
&=\sum_{M=-\ell}^{\ell}\langle \ell M|(j+k)m,j(M-m)\rangle\langle \ell M|(j'+k')m',j'(M-m')\rangle \underbrace{\frac{1}{4\pi}\int_{S^2}\overline{Y^j_{M-m}}Y^{j'}_{M-m'}}_{\frac{\delta_{j,j'}\delta_{m,m'}}{2j+1}}\\
&=\frac{\delta_{j,j'}\delta_{m,m'}}{2j+1}\underbrace{\sum_{M=-\ell}^{\ell}\langle \ell M|(j+k)m,j(M-m)\rangle\langle \ell M|(j+k')m,j(M-m)\rangle}_{\frac{2\ell+1}{2(j+k)+1}\delta_{(j+k),(j+k')}}\\
&=\delta_{j,j'}\delta_{k,k'}\delta_{m,m'}\frac{1}{2(j+k)+1}\frac{2\ell+1}{2j+1}
\end{aligned}$$

In line 2 we use the orthogonality of the Racah-normalized spherical harmonics. In the third line we use the orthogonality relation for the Clebsch-Gordan coefficients given in (19).

Secondly, we want to show that the expansion of a spherical tensor field $\mathbf{f} \in \mathscr{T}_\ell$ in terms of tensorial harmonics is unique and complete. Everybody agrees that the expansion of the individual components $(\mathbf{e}^\ell_M)^\top \mathbf{f}$ in spherical harmonics is complete. That is, we can write the expansion as

$$(\mathbf{e}^\ell_M)^\top \mathbf{f}(\mathbf{r}) = \sum_{j=0}^{\infty}\sum_{n=-j}^{j}\mathbf{b}^j_M(r)^\top \mathbf{Y}^j(\mathbf{r}),$$

where $\mathbf{b}^j_M(r) \in V_j$ are the expansion coefficients for the Mth component. We show the completeness of the tensorial harmonics by connecting them in an one-to-one manner with this ordinary spherical harmonic expansion of the spherical tensor field. For convenience we just consider the jth term in the expansion, i.e. the homogeneous part of $\mathbf{f}$ of order j that we denote by $\mathbf{f}^j$. We start with the expansion in terms of tensorial harmonics and rewrite them to identify the elements of $\mathbf{b}^j_M(r)$ written as $b^j_{M,n}(r)$ in terms of the $a^j_{km}(r)$. And so,

$$
\begin{aligned}
(\mathbf{e}^{\ell}_{M})^{\top}\mathbf{f}^{j}(\mathbf{r}) &= \sum_{k=-\ell}^{\ell}\sum_{m+n=M} a^{j}_{km}(r)\langle \ell M|(j+k)m, jn\rangle Y^{j}_{n}(\mathbf{r}) \\
&= \sum_{n=-j}^{j} Y^{j}_{n}(\mathbf{r}) \underbrace{\sum_{k=-\ell}^{\ell}\sum_{m} a^{j}_{km}(r)\langle \ell M|(j+k)m, jn\rangle}_{b^{j}_{M,n}(r)} \\
&= \sum_{n=-j}^{j} b^{j}_{M,n}(r) Y^{j}_{n}(\mathbf{r}).
\end{aligned}
$$

Now, we just have to give the inverse relation that computes the a^{j}_{km} out of the b^{j}_{Mn}. This can be accomplished by

$$
\begin{aligned}
&\sum_{M,n} b^{j}_{M,n}(r)\langle \ell M|(j+k')m', jn\rangle \\
&\qquad = \sum_{M,n}\sum_{k=-\ell}^{\ell}\sum_{m} a^{j}_{km}(r)\langle \ell M|(j+k)m, jn\rangle\langle \ell M|(j+k')m', jn\rangle \\
&\qquad = \sum_{k=-\ell}^{\ell}\sum_{m} a^{j}_{km}(r)\underbrace{\sum_{M,n}\langle \ell M|(j+k)m, jn\rangle\langle \ell M|(j+k')m', jn\rangle}_{\delta_{k,k'}\delta_{m,m'}\frac{2\ell+1}{2(j+k')+1}} \\
&\qquad = \frac{2\ell+1}{2(j+k')+1} a^{j}_{k'm'}(r),
\end{aligned}
$$

where we used again the orthogonality relation for the Clebsch-Gordan coefficients given in (19). This provides the one-to-one relation between the tensorial harmonic expansion with the component-wise spherical harmonic expansion and proves the statement.

3.1 Symmetric Tensor Fields

Typical filter kernels show certain symmetry properties. We figured out three symmetries that let vanish specific terms in the tensorial expansion: the rotationally symmetry with respect to a certain axis, the absence of torsion and reflection symmetry.

The rotation symmetry of a spherical tensor field $\mathbf{f} \in \mathscr{T}_\ell$ about the z-axis is expressed algebraically by the fact that $g_\phi \mathbf{f} = \mathbf{f}$ for all rotation g_ϕ around the z-axis. Such fields can easily be obtained by averaging a general tensor field $\mathbf{f}$ over all these rotations

$$
\mathbf{f}_{\mathrm{s}} = \frac{1}{2\pi}\int_0^{2\pi} g_\phi \mathbf{f}\, d\phi .
$$

It is well known that the representation $\mathbf{D}^j_{g_\phi}$ of such a rotation is diagonal, namely $D^j_{g_\phi,mm'} = \delta_{mm'} e^{\mathbf{i}m\phi}$. Hence, the expansion coefficients a^j_{km} of $\mathbf{f}_\mathrm{s}$ vanish for all $m \neq 0$. Thus, we can write any rotation symmetric tensor field as

$$\mathbf{f}_\mathrm{s}(\mathbf{r}) = \sum_{j=0}^{\infty} \sum_{k=-\ell}^{k=\ell} a^j_k(r)\; \mathbf{e}_0^{j+k} \circ_\ell \mathbf{Y}^j(\mathbf{r}). \tag{5}$$

We call such a rotation symmetric field torsion-free if $g_{yz}\mathbf{f}_\mathrm{s} = \mathbf{f}_\mathrm{s}$, where $g_{yz} \in O(3)$ is a reflection with respect to the *yz*-plane (or *xz*-plane). In Figure 1 we give an example of such a field. The action of such a reflection on spherical tensors is given by $D^j_{g_{yz},mm'} = (-1)^m \delta_{m(-m')}$. Similar to the rotational symmetry we can obtain such fields by averaging over the symmetry operation

$$\mathbf{f}_\mathrm{stf} = \frac{1}{2}(\mathbf{f}_\mathrm{s} + g_{yz}\mathbf{f}_\mathrm{s}).$$

Note, that the mirroring operation for a spherical harmonic is just a complex conjugation, that is $\mathbf{Y}^j(\mathbf{U}^T_{g_{yz}}\mathbf{r}) = \overline{\mathbf{Y}^j}(\mathbf{r})$. The consequence for equation (5) is that all terms where the $k+\ell$ are odd vanish. The reason for that is mainly Proposition 2.4 because with its help we can show that

$$\mathbf{D}^\ell_{g_{yz}}(\mathbf{e}_0^{j+k} \circ_\ell \mathbf{Y}^j(\mathbf{U}^T_{g_{yz}}\mathbf{r})) = (-1)^{(k+\ell)}(\mathbf{e}_0^{j+k} \circ_\ell \mathbf{Y}^j(\mathbf{r}))$$

holds.

Finally, consider the reflection symmetry with respect to the *xy*-plane. This symmetry is particularly important for rank 2 spherical tensor fields. In TV such fields are typically aligned or 'steered' with quantities of the same, even rank. For even rank tensors the parity of the underlying quantity is getting lost, so the voting field has to invariant under such parity changes. This symmetry is algebraically expressed by $g_{xy}\mathbf{f}_s = \mathbf{f}_s$ where $g_{xy} \in O(3)$ is a reflection with respect to the *xy*-plane, whose action on spherical tensors is given by $D^j_{g_{yz},mm'} = (-1)^j \delta_{mm'}$. Averaging over this symmetry operation has the consequence that expansion terms with odd j are vanishing. For odd rank tensor fields the reflection symmetry is not imperative. But there is typically an antisymmetry of the form $g_{xy}\mathbf{f}_s = -\mathbf{f}_s$. This antisymmetry let vanish the expansion terms with even index j.

3.2 Expanding Rotation-Symmetric Fields in Polar Representation

We write the spherical tensor field in polar representation $\mathbf{f}(r,\theta,\phi)$, where $\cos(\theta) = z/r$ and $\phi = \arg(x+\mathbf{i}y)$. Consider a field of rank ℓ. In polar representation the rotation symmetry with respect to the *z*-axis is expressed by the fact that for all $m = -\ell,\ldots,\ell$ the components $f_m(r,\theta,\phi)$ of the field $\mathbf{f}$ can be written as

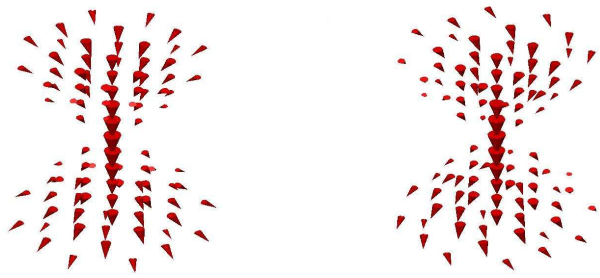

Fig. 1 Rotation symmetric vector fields. Left: torsion-free. Right: with torsion.

$$f_m(r,\theta,\phi) = \alpha_m(r,\theta)e^{\mathbf{i}m\phi},$$

where $\alpha_m(r,\theta) \in \mathbb{C}$ is the colatitudinal/radial dependency of the field. This rotation symmetry is easy to verify because $f_m(r,\theta,\phi-\phi')e^{\mathbf{i}m\phi'} = f_m(r,\theta,\phi)$. For torsion-free tensor fields we additionally know that $\alpha_m(r,\theta) \in \mathbb{R}$. To project such a symmetric kind of field on the tensorial harmonics consider the mth component of the tensorial harmonic $\mathbf{Z}^j_{k0}$:

$$\begin{aligned}
(\mathbf{e}^\ell_m)^\top \mathbf{Z}^j_{k0}(\theta,\phi) &= (\mathbf{e}^\ell_m)^\top (\mathbf{e}^{j+k}_0 \circ_\ell \mathbf{Y}^j(\theta,\phi)) \\
&= \langle \ell m \mid (j+k)0, jm\rangle Y^j_m(\theta,\phi) \\
&= \langle \ell m \mid (j+k)0, jm\rangle e^{\mathbf{i}m\phi}\sqrt{\frac{(j-m)!}{(j+m)!}}P^j_m(\cos(\theta)) \\
&= C_{\ell jm} e^{\mathbf{i}m\phi} P^j_m(\cos(\theta))
\end{aligned}$$

Now, using this expression the projection on $\mathbf{Z}^j_{k0}$ yields

$$\begin{aligned}
\langle \mathbf{Z}^j_{k0}, \mathbf{f}\rangle_{S^2} &= \int_{-\pi/2}^{\pi/2}\int_0^{2\pi} \mathbf{Z}^j_{k0}(\theta,\phi)^\top \mathbf{f}(r,\theta,\phi)\sin(\theta)\, d\phi d\theta \\
&= 2\pi \sum_{m=-\ell}^{\ell} C_{\ell jm} \int_{-\pi/2}^{\pi/2} \alpha_m(r,\theta) P^j_m(\cos(\theta))\sin(\theta) d\theta
\end{aligned}$$

The residue integral may be computed numerically or analytically.

3.3 Rotational Steering

By equation (4) the tensorial harmonics are very well suited to rotate the expanded spherical tensor field. We want to show how to steer a rotation symmetric field efficiently in a certain direction.

Consider a general rotation $g_{\mathbf{n}} \in SO(3)$ that rotates the z-axis $\mathbf{r}_z = (0,0,1)^\top$ to some given orientation $\mathbf{n} \in \mathbb{R}^3$, i.e. $\mathbf{R}_{g_{\mathbf{n}}}\mathbf{r}_z = \mathbf{n}$. Of course, there are several rotations that can accomplish this. But, if we apply such a rotation on a rotational symmetric field $\mathbf{f}_\mathrm{s}$ this additional freedom does not have an influence on the result. Starting from the general rotation behavior of the tensorial harmonic expansion in eq. (4) one can derive that the symmetric tensor field $\mathbf{f}_\mathrm{s}$ rotates as

$$(g_{\mathbf{n}}\mathbf{f}_\mathrm{s})(\mathbf{r}) = \sum_{j=0}^{\infty} \sum_{k=-\ell}^{k=\ell} a_k^j(r)\ \mathbf{Y}^{j+k}(\mathbf{n}) \circ_\ell \mathbf{Y}^j(\mathbf{r}) \tag{6}$$

This expression is the basis for the algorithm proposed in the next section. To prove equation (6) one needs to know that $\mathbf{Y}^j(\mathbf{r}_z) = \mathbf{e}_0^j$.

4 Local Adaptive Filtering with Tensorial Harmonics

We already described the two dual ideas of local adaptive filtering in the introduction. In this Section we describe how tensorial harmonics can be used to compute the filter integrals efficiently. For both cased we assume that the filter kernel is tensor-valued of rank ℓ, i.e. a function $\mathbf{V}^n : \mathbb{R}^3 \to V_\ell$. The intensity image is still represented by the function $m : \mathbb{R}^3 \to \mathbb{R}$ and an orientation image $\mathbf{n} : \mathbb{R}^3 \to V_1$ of normalized vectors is given. We also assume a rotation symmetric filter kernel as given in equation (6). The expansion coefficients $a_k^j(r)$ can be obtained by a projection of the filter kernel on the tensorial harmonics

$$a_k^j(r) = N_{j,k} \langle \mathbf{Z}_{k0}^j, \mathbf{V}^{\mathbf{r}_z} \rangle_{S_r^2} \tag{7}$$

due to the symmetry only $\mathbf{Z}_{k0}^j$ are involved. For the numerical integration scheme Section 3.2.

4.1 The Convolution Integral

The key expression that has to be computed is

$$\mathbf{U}_\mathrm{conv}(\mathbf{r}) = \int_{\mathbb{R}^3} \mathbf{V}^{\mathbf{n}(\mathbf{r}')}(\mathbf{r} - \mathbf{r}')\ m(\mathbf{r}')\ d\mathbf{r}', \tag{8}$$

Following the last section we set the voting field to $\mathbf{V}^{\mathbf{n}}(\mathbf{r}) = (g_{\mathbf{n}}\mathbf{f}_s)(\mathbf{r})$, where $\mathbf{f}_s$ is the rotational symmetric field. Inserting this expression in (8) and using eq. (6) yields

$$\begin{aligned}
\mathbf{U}_{\text{conv}}(\mathbf{r}) &= \int_{\mathbb{R}^3} \mathbf{V}^{\mathbf{n}(\mathbf{r}')}(\mathbf{r}-\mathbf{r}')\, m(\mathbf{r}')\, d\mathbf{r}' = \int_{\mathbb{R}^3} (g_{\mathbf{n}(\mathbf{r}')}\mathbf{f}_s)(\mathbf{r}-\mathbf{r}')\, m(\mathbf{r}')\, d\mathbf{r}' \\
&= \int_{\mathbb{R}^3} \sum_{j=0}^{\infty} \sum_{k=-\ell}^{k=\ell} a_k^j(|\mathbf{r}-\mathbf{r}'|)\, \mathbf{Y}^{j+k}(\mathbf{n}(\mathbf{r}')) \circ_\ell \mathbf{Y}^j(\mathbf{r}-\mathbf{r}')\, m(\mathbf{r}')\, d\mathbf{r}' \\
&= \sum_{j=0}^{\infty} \sum_{k=-\ell}^{k=\ell} \int_{\mathbb{R}^3} \underbrace{m(\mathbf{r}')\mathbf{Y}^{j+k}(\mathbf{n}(\mathbf{r}'))}_{\mathbf{E}^{j+k}(\mathbf{r}')} \circ_\ell \underbrace{a_k^j(|\mathbf{r}-\mathbf{r}'|)\mathbf{Y}^j(\mathbf{r}-\mathbf{r}')}_{\mathbf{A}_k^j(\mathbf{r}-\mathbf{r}'))}\, d\mathbf{r}' \\
&= \sum_{j=0}^{\infty} \sum_{k=-\ell}^{k=\ell} \mathbf{E}^{j+k} \tilde{\circ}_\ell \mathbf{A}_k^j
\end{aligned}$$

where $\mathbf{E}^j(\mathbf{r}) := m(\mathbf{r})\mathbf{Y}^j(\mathbf{n}(\mathbf{r}))$ are combined tensor-valued evidence images and $\mathbf{A}_k^j(\mathbf{r}) := a_k^j(r)\mathbf{Y}^j(\mathbf{r})$ is the harmonic expansion of the voting field. In Algorithm 1 we give pseudo-code for implementation.

Algorithm 1 Convolution Algorithm

Input: $m \in \mathscr{T}_0$, $\mathbf{n}(\mathbf{r}) \in \mathscr{T}_1$, $\mathbf{A}_k^j \in \mathscr{T}_j$
Output: $\mathbf{U} \in \mathscr{T}_\ell$

```
Let E^0 := m
for j = 1 : (j_max + ℓ)  do
    E^j := (E^{j-1} ∘_j n)/⟨j0|10,(j-1)0⟩
end for
for j = 0 : j_max  do
    for k = -ℓ : 2 : ℓ  do
        Compute U := U + E^{j+k} ∘̃_ℓ A_k^j
    end for
end for
```

4.2 The Correlation Integral

Let us now consider the correlation integral

$$\mathbf{U}_{\text{corr}}(\mathbf{r}) = \int_{\mathbb{R}^3} \mathbf{V}^{\mathbf{n}(\mathbf{r})}(\mathbf{r}'-\mathbf{r})\, m(\mathbf{r}')\, d\mathbf{r}'. \tag{9}$$

Following the same approach as in the previous section we can write

Algorithm 2 Correlation Algorithm

Input: $m \in \mathscr{T}_0$, $\mathbf{n}(\mathbf{r}) \in \mathscr{T}_1$, $\mathbf{A}_k^j \in \mathscr{T}_j$
Output: $\mathbf{U} \in \mathscr{T}_\ell$
Let $\mathbf{N}^0 := 1$
for $j = 1 : (j_{\max} + \ell)$ **do**
  $\mathbf{N}^j := (\mathbf{N}^{j-1} \circ_j \mathbf{n}) / \langle j0|10, (j-1)0 \rangle$
end for
for $j = 0 : j_{\max}$ **do**
  for $k = -\ell : 2 : \ell$ **do**
    Compute $\mathbf{U} := \mathbf{U} + \mathbf{N}^{j+k} \circ_\ell (m * \mathbf{A}_k^j)$
  end for
end for

$$
\begin{aligned}
\mathbf{U}_{\text{corr}}(\mathbf{r}) &= \int_{\mathbb{R}^3} \sum_{j=0}^{\infty} \sum_{k=-\ell}^{k=\ell} a_k^j(|\mathbf{r}' - \mathbf{r}|)\, \mathbf{Y}^{j+k}(\mathbf{n}(\mathbf{r})) \circ_\ell \mathbf{Y}^j(\mathbf{r}' - \mathbf{r})\, m(\mathbf{r}')\, d\mathbf{r}' \\
&= \sum_{j=0}^{\infty} \sum_{k=-\ell}^{k=\ell} \underbrace{\mathbf{Y}^{j+k}(\mathbf{n}(\mathbf{r}))}_{\mathbf{N}^{j+k}(\mathbf{r})} \int_{\mathbb{R}^3} m(\mathbf{r}') \circ_\ell \underbrace{a_k^j(|\mathbf{r}' - \mathbf{r}|)\mathbf{Y}^j(\mathbf{r}' - \mathbf{r})}_{\mathbf{A}_k^j(\mathbf{r}' - \mathbf{r}))}\, d\mathbf{r}' \\
&= \sum_{j=0}^{\infty} \sum_{k=-\ell}^{k=\ell} \mathbf{N}^{j+k} \circ_\ell (m * \mathbf{A}_k^j)
\end{aligned}
$$

The final expression enables us to give an efficient computation scheme as depicted in Algorithm 2.

5 Spherical Tensor Derivatives

In this Section we propose derivative operators that connects spherical tensor fields of different ranks. We call them spherical tensor derivatives (STD). They can be used to compute local adaptive filters for special types of filter kernels more efficiently. The explicit convolutions are replaced by finite difference operations.

They idea is to represent the filter kernel by superpositions of STDs of radial symmetric functions. Due to the commuting property of convolution and differentiation the computation of the filter response will just involve one explicit convolution with the radial symmetric functions, the rest of the computations consists of repeated applications of STDs.

In particular we will consider spherical derivatives of the Gaussian. We will see that the resulting polynomials are just solid harmonics (see Section 2.4). Based on this we will present a special type of filter kernel which can be defined for arbitrary tensor ranks and has a very simple parameter dependency controlling its shape and orientation.

Proposition 5.1 (Spherical Tensor Derivatives) *Let* $\mathbf{f} \in \mathscr{T}_\ell$ *be a tensor field. The spherical up-derivative* $\nabla^1 : \mathscr{T}_\ell \to \mathscr{T}_{\ell+1}$ *and the down-derivative* $\nabla_1 : \mathscr{T}_\ell \to \mathscr{T}_{\ell-1}$ *are defined as*

$$\nabla^1 \mathbf{f} := \nabla \bullet_{\ell+1} \mathbf{f} \tag{10}$$

$$\nabla_1 \mathbf{f} := \nabla \bullet_{\ell-1} \mathbf{f}, \tag{11}$$

where

$$\nabla = \left(\frac{1}{\sqrt{2}}(\partial_x - \mathbf{i}\partial_y), \partial_z, -\frac{1}{\sqrt{2}}(\partial_x + \mathbf{i}\partial_y)\right)$$

is the spherical gradient and $\partial_x, \partial_y, \partial_z$ *the standard partial derivatives.*

Proof. We have to show that $\nabla^1 \mathbf{f} \in \mathscr{T}_{\ell+1}$, i.e.

$$\nabla^1(\mathbf{D}_g^\ell \mathbf{f}(\mathbf{U}_g^T \mathbf{r})) = \mathbf{D}_g^{\ell+1}(\nabla^1 \mathbf{f})(\mathbf{U}_g^T \mathbf{r})$$

and $\nabla_1 \mathbf{f} \in \mathscr{T}_{\ell-1}$

$$\nabla_1(\mathbf{D}_g^\ell \mathbf{f}(\mathbf{U}_g^T \mathbf{r})) = \mathbf{D}_g^{\ell-1}(\nabla_1 \mathbf{f})(\mathbf{U}_g^T \mathbf{r})$$

Both statements are proved just by using the properties of $\bullet$.

Note, that for a scalar function the spherical up-derivative is just the spherical gradient, i.e. $\nabla f = \nabla^1 f$.

In the Fourier domain the spherical derivatives act by point-wise $\bullet$-multiplications with a solid harmonic $\mathbf{i}k\mathbf{Y}^1(\mathbf{k}) = \mathbf{i}\mathbf{R}^1(\mathbf{k}) = \mathbf{i}\mathbf{S}\mathbf{k}$ where $k = \|\mathbf{k}\|$ the frequency magnitude:

Proposition 5.2 (Fourier Representation) *Let* $\widetilde{\mathbf{f}}(\mathbf{k})$ *be the Fourier transformation of some* $\mathbf{f} \in \mathscr{T}_\ell$ *and* $\widetilde{\nabla}$ *representations of the spherical derivative in the Fourier domain that are implicitly defined by* $\widetilde{(\nabla \mathbf{f})} = \widetilde{\nabla}\widetilde{\mathbf{f}}$*, then*

$$\widetilde{\nabla}^1 \widetilde{\mathbf{f}}(\mathbf{k}) = \mathbf{R}^1(\mathbf{i}\mathbf{k}) \bullet_{\ell+1} \widetilde{\mathbf{f}}(\mathbf{k}) \tag{12}$$

$$\widetilde{\nabla}_1 \widetilde{\mathbf{f}}(\mathbf{k}) = \mathbf{R}^1(\mathbf{i}\mathbf{k}) \bullet_{\ell-1} \widetilde{\mathbf{f}}(\mathbf{k}). \tag{13}$$

Proof. By the ordinary Fourier correspondence for the partial derivative, namely $\widetilde{\partial_x \mathbf{f}} = \mathbf{i}k_x \widetilde{\mathbf{f}}$, we can verify for the spherical gradient ∇ that

$$\widetilde{\nabla} = \mathbf{i}\mathbf{S}\mathbf{k} = \mathbf{R}^1(\mathbf{i}\mathbf{k})$$

and hence

$$\widetilde{\nabla^1 \mathbf{f}} = \widetilde{(\nabla \bullet_{\ell+1} \mathbf{f})} = \widetilde{\nabla} \bullet_{\ell+1} \widetilde{\mathbf{f}} = \mathbf{R}^1(\mathbf{i}\mathbf{k}) \bullet_{\ell+1} \widetilde{\mathbf{f}}$$

which was to show. Proceed similar for the down-derivative.

In the following we want to use as a short-hand notation for multiple STDs

$$\nabla_i^\ell := \nabla_i \nabla^\ell := \underbrace{\nabla_1 \ldots \nabla_1}_{i-\text{times}} \underbrace{\nabla^1 \ldots \nabla^1}_{\ell-\text{times}}.$$

which we immediately use in this

Proposition 5.3 (Commuting Property for Convolutions) *Let* $\mathbf{A} \in \mathscr{T}_k$ *and* $\mathbf{B} \in \mathscr{T}_j$ *be arbitrary spherical tensor fields then*

$$(\nabla^\ell \mathbf{A}) \,\widetilde{\bullet}_J\, \mathbf{B} = \mathbf{A} \,\widetilde{\bullet}_J (\nabla_\ell \mathbf{B}) \tag{14}$$

$$(\nabla^\ell \mathbf{A}) \,\widetilde{\bullet}_L\, \mathbf{B} = \mathbf{A} \,\widetilde{\bullet}_L (\nabla^\ell \mathbf{B}) \tag{15}$$

where $J = j - (\ell + k)$ *and* $L = j + \ell + k$.

Proof. Both assertions are founded by the associativity of the spherical product. Consider the first statement in the Fourier domain by using equation (12) and then apply the associativity given in equation (2):

$$\begin{aligned}(\widetilde{\nabla^\ell}\widetilde{\mathbf{A}}) \bullet_J \widetilde{\mathbf{B}} &= (\mathbf{R}^1 \bullet_{k+\ell} (\widetilde{\nabla^{\ell-1}}\widetilde{\mathbf{A}})) \bullet_J \widetilde{\mathbf{B}} \\ &= (\widetilde{\nabla^{\ell-1}}\widetilde{\mathbf{A}}) \bullet_J (\mathbf{R}^1 \bullet_{j-1} \widetilde{\mathbf{B}}) = (\widetilde{\nabla^{\ell-1}}\widetilde{\mathbf{A}}) \bullet_j (\widetilde{\nabla}_1 \widetilde{\mathbf{B}})\end{aligned}$$

where we abbreviated $\mathbf{R}^1 = \mathbf{R}^1(\mathbf{ik})$. A repeated application of this proves the first assertion. For the second statement it is similar but using the associativity as given in equation (1) .

5.1 Spherical Gaussian Derivatives

Our goal is to represent filter kernels as linear combinations of STDs of radial symmetric functions. Suppose that g is an arbitrary radial functions, i.e. $g(\mathbf{r}) = g(\|\mathbf{r}\|)$. In fact, it holds in general that the angular part of STDs of the form $\nabla_i^n g$ are spherical harmonics of degree $n - i$. In particular we are interested in a very important radial function, the Gaussian function. In this section we show that the STDs of a Gaussian are just the Gaussian-windowed solid harmonics.

Proposition 5.4 *The Gaussian windowed harmonic of width* σ *is defined as*

$$\mathbf{V}_\sigma^\ell(\mathbf{r}) := \frac{1}{\sigma^3}\left(\frac{-r}{\sigma^2}\right)^\ell \mathbf{Y}^\ell(\mathbf{r}) e^{-\frac{r^2}{2\sigma^2}},$$

then, the Fourier transformation of $\mathbf{V}^\ell(\mathbf{r})$ *is given by*

$$\tilde{\mathbf{V}}_\sigma^\ell(\mathbf{k}) = \langle e^{\mathbf{ik}^\top \mathbf{r}}, \mathbf{V}^\ell(\mathbf{r})\rangle_{L_2} = (\mathbf{i}k)^\ell \mathbf{Y}^\ell(\mathbf{k}) e^{-\frac{(\sigma k)^2}{2}}.$$

Proof. We start with the definition of the Fourier transform and plug in the spherical harmonic expansion of the plane wave in terms of spherical Bessel function j_n (see e.g [9], p. 136). Then, we integrate out the angular dependend part:

$$\int_{\mathbb{R}^3} \mathbf{V}^\ell_\sigma(\mathbf{r}) e^{-\mathbf{i}\mathbf{k}^\top \mathbf{r}} d\mathbf{r} = \int_{\mathbb{R}^3} \mathbf{V}^\ell_\sigma(\mathbf{r}) \sum_n (2n+1)(-\mathbf{i})^n \; j_n(kr) \; \mathbf{Y}^n(\mathbf{r}) \bullet_0 \mathbf{Y}^n(\mathbf{k})$$
$$= \frac{(\mathbf{i})^\ell}{\sigma^{2\ell+3}} \mathbf{Y}^\ell(\mathbf{k}) \int_0^\infty j_\ell(kr) e^{-\frac{r^2}{2\sigma^2}} r^{\ell+2} dr$$

The residual radius dependend part is integrated by using the series expansion of the spherical Bessel function:

$$\int_0^\infty j_\ell(kr) e^{-\frac{r^2}{2\sigma^2}} r^{2+\ell} dr = \sum_{n=0}^\infty \frac{(-1)^n k^{2n+\ell}}{2^n n!(2(n+\ell)+1))!!} \underbrace{\int_0^\infty r^{2n+2\ell+2} e^{-\frac{r^2}{2\sigma^2}} dr}_{\sigma^{2(n+\ell)+3}(2(n+\ell)+1)!!}$$
$$= \sum_{n=0}^\infty \frac{(-1)^n k^{2n+\ell}}{2^n n! \sigma^{-2(n+\ell)-3}} = \sigma^{2\ell+3} k^\ell e^{-\frac{(\sigma k)^2}{2}}$$

So we arrive at

$$\tilde{\mathbf{V}}^\ell_\sigma(\mathbf{k}) = \int_{\mathbb{R}^3} \mathbf{V}^\ell_\sigma(\mathbf{r}) e^{-\mathbf{i}\mathbf{k}^\top \mathbf{r}} d\mathbf{r} = (\mathbf{i}k)^\ell \mathbf{Y}^\ell(\mathbf{k}) e^{-\frac{(\sigma k)^2}{2}}$$

which proves the assertion.

In fact, for $\sigma = 1$ the $\mathbf{V}^\ell$s are eigenfunctions of the Fourier transformation with eigenvalue $(-\mathbf{i})^\ell$. Using the above proposition it is also easy to show that the $\mathbf{V}^\ell$ are just the ℓth order spherical derivatives of a Gaussian.

Proposition 5.5 (Spherical Gaussian Derivative) *The homogeneous spherical derivative* ∇^ℓ *of a Gaussian computes to*

$$\nabla^\ell e^{-\frac{r^2}{2\sigma^2}} = \sigma^3 \mathbf{V}^\ell_\sigma(\mathbf{r}) = \left(-\frac{1}{\sigma^2}\right)^\ell \mathbf{R}^\ell(\mathbf{r}) \; e^{-\frac{r^2}{2\sigma^2}}$$

Proof. An immediate consequence of the fact that $\widetilde{\nabla^\ell} \widetilde{g}(\mathbf{k}) = \mathbf{R}^\ell(\mathbf{i}\mathbf{k}) \widetilde{g}(\mathbf{k})$ and Proposition 5.4.

6 Local Adaptive Filtering with STDs

The basic idea of the following approach is to represent the filter kernel by a linear superposition of spherical Gaussian derivatives. This will enable us to formulate the filtering process by repeated applications of spherical derivatives which is much more efficient than the explicit convolutions used in the previous section. We have seen that the Gaussian derivatives are just Gaussian-windowed harmonic polynomials, so the resulting kernels will be also Gaussian windowed harmonics. There are many possibilities to construct such filter kernels. We present a kernel which can be imagined as a squeezed or stretched Gaussian. But actually, we restrict the expansion to derivatives of the form $\nabla^j g$. Due to the symmetry properties of the Gaussian the expansion will only contain even degree derivatives $\nabla^{2j} g$. We propose to use the following filter kernel

$$\mathbf{V}^{\mathbf{n}}(\mathbf{r}) = \sum_{j=0}^{\infty} \frac{\lambda^j}{(2j-1)!!} \mathbf{R}^{2j+\ell}(\mathbf{n}) \bullet_\ell \nabla^{2j} g(\mathbf{r}) \tag{16}$$

where $\mathbf{n}$ is the squeezing/stretching direction. The expression $(2j-1)!!$ denotes the double factorial given by $(2n-1)(2n-3)\ldots 3$. The parameter $\ell \geq 0$ determines the rank of the filter kernel. The parameter λ controls the shape. For $\lambda < 0$ the function has a tablet-like shape, for $\lambda > 0$ the shape is stick-like. Note that, the if the orientation parameter $\mathbf{n}$ is not normalized the magnitude $\|\mathbf{n}\|$ has the same effect on the shape of the filter kernel like the magnitude of λ. So, we can control the orientation as well as the shape of the filter kernel by the single parameter $\mathbf{n}$. In Figure 2 we show surface plots of the filter kernel for $\ell = 0$ for different λ.

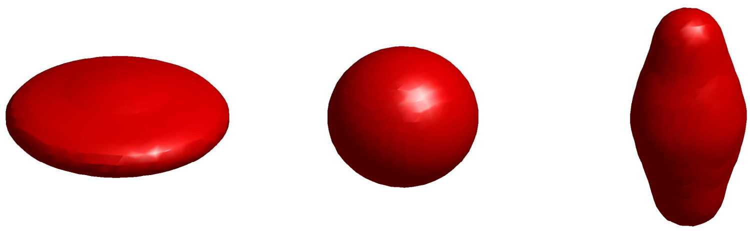

Fig. 2 Surface-Plots for $\ell = 0$ with $\lambda = -0.2, 0, 0.2$

6.1 *The Convolution Integral*

Again we have to compute the convolution integral as given in equation (8). Inserting the filter kernel as given in equation (16) into (8) yields

$$\begin{aligned}
\mathbf{U}_{\text{conv}}(\mathbf{r}) &= \int_{\mathbb{R}^3} \sum_{j=0}^{\infty} \frac{\lambda^j}{(2j-1)!!} \underbrace{m(\mathbf{r}')\mathbf{R}^{2j+\ell}(\mathbf{n}(\mathbf{r}'))}_{\mathbf{E}^j(\mathbf{r}')} \bullet_\ell (\nabla^{2j} g)(\mathbf{r}-\mathbf{r}') d\mathbf{r}' \\
&= \sum_{j=0}^{\infty} \frac{\lambda^j}{(2j-1)!!} (\nabla^{2j} g) \tilde{\bullet}_\ell \mathbf{E}^j \\
&= g * \sum_{j=0}^{\infty} \frac{\lambda^j}{(2j-1)!!} \nabla_{2j} \mathbf{E}^j
\end{aligned}$$

where we used equation (14) to get from the second to the third line. The resulting approach is similar to the algorithm using the tensorial harmonics, but the convolutions with the basis functions are replaced by repeated differentiations.

Algorithm 3 Convolution Algorithm with STDs

Input: $m \in \mathcal{T}_0$, $\mathbf{n}(\mathbf{r}) \in \mathcal{T}_1$
Output: $\mathbf{U} \in \mathcal{T}_\ell$

Let $\mathbf{E}^0 := m \bullet_\ell \mathbf{R}^\ell(\mathbf{n})$
for $j = 1 : j_{\max}$ **do**
  $\mathbf{E}^j := \mathbf{E}^{j-1} \bullet_{2j+\ell} \mathbf{R}^2(\mathbf{n})$
end for
Let $\mathbf{U} := 0$
for $j = j_{\max} : -1 : 1$ **do**
  $\mathbf{U} := \frac{\lambda^j}{(2j-1)!!} \nabla_2(\mathbf{U} + \mathbf{E}^j)$
end for
$\mathbf{U} := \mathbf{U} + \mathbf{E}^0$
$\mathbf{U} := g * \mathbf{U}$

6.2 The Correlation Integral

On the other hand consider the correlation integral. Starting with equation (9) and inserting expression (16) yields:

$$\mathbf{U}_{\mathrm{corr}}(\mathbf{r}) = \int_{\mathbb{R}^3} \sum_{j=0}^{\infty} \frac{\lambda^j}{(2j-1)!!} \underbrace{\mathbf{R}^{2j+\ell}(\mathbf{n}(\mathbf{r}))}_{\mathbf{N}^j(\mathbf{r})} \bullet_\ell (\nabla^{2j} g)(\mathbf{r}' - \mathbf{r}) m(\mathbf{r}')\, d\mathbf{r}'$$
$$= \sum_{j=0}^{\infty} \frac{\lambda^j}{(2j-1)!!} \mathbf{N}^j(\mathbf{r}) \bullet_\ell \nabla^{2j}(m * g)$$

where we used equation (15) to pull the differentiation outward. In Algorithm 4 we depict the computation process. Comparing to the convolution integral everything can be computed in place. We just need one loop for the whole process, hence, the memory consumption is much lower as for the convolution algorithm.

6.3 Application to Anisotropic Blurring

Finally we want to use the algorithm proposed in the last section to denoise scalar MRI data while preserving edges and surfaces, that is, we apply Algorithm 4 with $\ell = 0$. The idea is to perform a blurring operation isotropically in isotropic regions

Algorithm 4 Correlation Algorithm with STDs

Input: $m \in \mathscr{T}_0$, $\mathbf{n}(\mathbf{r}) \in \mathscr{T}_1$
Output: $\mathbf{U} \in \mathscr{T}_\ell$
Let $\mathbf{N} := \mathbf{R}^\ell(\mathbf{n})$
Let $\mathbf{M} := m * g$
Let $\mathbf{U} := \mathbf{N} \bullet_\ell \mathbf{M}$
for $j = 1 : j_{\max}$ **do**
  $\mathbf{N} := \mathbf{N} \bullet_{2j+\ell} \mathbf{R}^2(\mathbf{n})$
  $\mathbf{M} := \nabla^2 \mathbf{M}$
  $\mathbf{U} := \mathbf{U} + \frac{\lambda^j}{(2j-1)!!} \mathbf{N} \bullet_\ell \mathbf{M}$
end for

and anisotropically in anisotropic regions. As a measure anisotropy we use the gradient normalized with the local standard deviation. We choose $\lambda < 0$ such that the filter kernel has a tablet-like shape. This tablet-shape is for each voxel oriented orthogonal to the observed gradient such that the smoothing is not performed across the edges. In conclusion we choose the orientation/shape parameter **n** as

$$\mathbf{n} = \frac{\nabla^1(m * g)}{\varepsilon + \sqrt{m^2 * g - (m * g)^2}}$$

where $\varepsilon > 0$ is a small regularization parameter avoiding zero divisions. In Figure 3 we show an example applied on MRI-data of a human head of size 256^3. Obviously, the algorithm works, the isotropic regions are smoothed well and the edges are kept. The running time is on a standard PC (Intel Pentium 2.2 Ghz) is about 15 seconds.

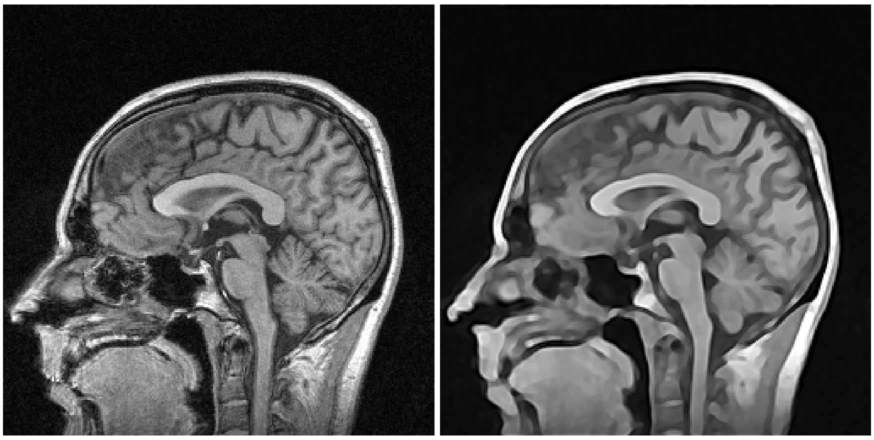

Fig. 3 Example of anisotropic blurring filter on MRI-data

Appendix

Spherical Harmonics

We always use Racah-normalized spherical harmonics. In terms of Legendre polynomials they are written as

$$Y_m^\ell(\phi,\theta) = \sqrt{\frac{(l-m)!}{(l+m)!}} P_m^\ell(\cos(\theta)) e^{\mathbf{i}\phi}$$

We always write $\mathbf{r} \in S^2$ instead of (ϕ,θ). The Racah-normalized solid harmonics can be written as

$$R_m^\ell(\mathbf{r}) = \sqrt{(\ell+m)!(\ell-m)!} \sum_{i,j,k} \frac{\delta_{i+j+k,\ell}\delta_{i-j,m}}{i!j!k!2^i 2^j} (x-\mathbf{i}y)^j (-x-\mathbf{i}y)^i z^k,$$

where $\mathbf{r} = (x,y,z)$. They are related to spherical harmonics by $R_m^\ell(\mathbf{r})/r^\ell = Y_m^\ell(\mathbf{r})$

Clebsch-Gordan Coefficients

For the computation of the Clebsch-Gordan (CG) coefficients recursive formulas are applied (see e.g. [1]). The important orthogonality-relations of the CG-coefficients are

$$\sum_{j,m} \langle jm|j_1 m_1, j_2 m_2\rangle \langle jm|j_1 m_1', j_2 m_2'\rangle = \delta_{m_1,m_1'}\delta_{m_2,m_2'} \tag{17}$$

$$\sum_{m=m_1+m_2} \langle jm|j_1 m_1, j_2 m_2\rangle \langle j'm'|j_1 m_1, j_2 m_2\rangle = \delta_{j,j'}\delta_{m,m'} \tag{18}$$

$$\sum_{m_1,m} \langle jm|j_1 m_1, j_2 m_2\rangle \langle jm|j_1 m_1, j_2' m_2'\rangle = \frac{2j+1}{2j_2'+1}\delta_{j_2,j_2'}\delta_{m_2,m_2'} \tag{19}$$

For two special cases there are explicit formulas:

$$\langle \ell m|(\ell-\lambda)(m-\mu), \lambda\mu\rangle = \binom{\ell+m}{\lambda+\mu}^{1/2} \binom{\ell-m}{\lambda-\mu}^{1/2} \binom{2\ell}{2\lambda}^{-1/2} \tag{20}$$

$$\langle \ell m|(\ell+\lambda)(m-\mu), \lambda\mu\rangle = (-1)^{\lambda+\mu} \binom{\ell+\lambda-m+\mu}{\lambda+\mu}^{1/2} \binom{\ell+\lambda+m-\mu}{\lambda-\mu}^{1/2} \binom{2\ell+2\lambda+1}{2\lambda}^{-1/2} \tag{21}$$

The CG-coefficients fulfill certain symmetry relations

$$\langle jm|j_1m_1,j_2m_2\rangle = \langle j_1m_1,j_2m_2|jm\rangle \tag{22}$$
$$\langle jm|j_1m_1,j_2m_2\rangle = (-1)^{j+j_1+j_2}\langle jm|j_2m_2,j_1m_1\rangle \tag{23}$$
$$\langle jm|j_1m_1,j_2m_2\rangle = (-1)^{j+j_1+j_2}\langle j(-m)|j_1(-m_1),j_2(-m_2)\rangle \tag{24}$$

Wigner D-Matrix

The components of $\mathbf{D}_g^\ell$ are written D_{mn}^ℓ. In Euler angles ϕ,θ,ψ in ZYZ-convention we have

$$D_{mn}^\ell(\phi,\theta,\psi) = e^{\mathbf{i}m\phi}d_{mn}^\ell(\theta)e^{\mathbf{i}n\psi},$$

where $d_{mn}^\ell(\theta)$ is the Wigner d-matrix which is real-valued. Explicit formulas for the $d_{mn}^\ell(\theta)$ involve the Jacobi-polynomials (see e.g. [9]) The important relations to the Clebsch-Gordan coefficients are:

$$D_{mn}^\ell = \sum_{\substack{m_1+m_2=m\\ n_1+n_2=n}} D_{m_1n_1}^{\ell_1}D_{m_2n_2}^{\ell_2}\langle lm|l_1m_1,l_2m_2\rangle\langle ln|l_1n_1,l_2n_2\rangle \tag{25}$$

and

$$D_{m_1n_1}^{\ell_1}D_{m_2n_2}^{\ell_2} = \sum_{l,m,n} D_{mn}^\ell\langle lm|l_1m_1,l_2m_2\rangle\langle ln|l_1n_1,l_2n_2\rangle. \tag{26}$$

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