MAXIMUM-A-POSTERIORI RESTORATION OF IMAGES -
AN APPLICATION OF THE VITERBI ALGORITHM TO TWO-DIMENSIONAL FILTERING

H. Burkhardt and H. Schorb

Institut für Mess- und Regelungstechnik
Universität (TH) Karlsruhe
D-7500 Karlsruhe 1, Federal Republic of Germany

Abstract

In many restoration problems, the a-priori knowledge of a finite number of pixel amplitudes of the original image is available (e.g. blurred black-and-white images). It is shown how to incorporate this information into optimal image reconstruction. The degradation of a discrete image is modeled as a two-dimensional, finite-state Markov process. Dynamic programming is then applied to get an optimal estimate of the state sequence of that process observed in memoryless noise, a technique which is known as Viterbi algorithm. This leads to a nonlinear recursive filter providing superior performance over optimal linear filtering. Examples are given in comparison with inverse filtering.

Introduction

In many papers and textbooks on image processing, one can find the restoration of degraded binary images with linear filters as the ultimate solution. However, it can easily be shown that linear filtering is non-optimal for this task because the a-priori knowledge of the binary valued original image cannot be incorporated into the solution.

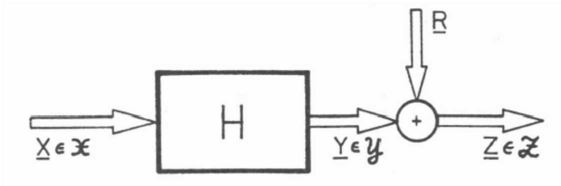

Fig. 1: Degradation model

The degradation of images can often be interpreted from the viewpoint of communication theory as sending source information in form of the original image $\underline{X}$ over a channel with spatial dispersion and additive noise (Fig. 1). If this source information belongs to a finite set of discrete signals

$$\underline{X} \in \mathcal{X} \ , \qquad (1.1)$$

then we talk about digital communication. In the case of pulse-amplitude-modulation (PAM), for example, discrete-time and discrete-valued sequences are sent through analog channels. The signal detector has to cope with problems like intersymbol interference, closure of eye-patterns, and ill-conditioned or singular transfer functions. The application of linear filters (like optimal transversal or KaIman filters) gives a poor performance improvement for channels with severe distortions. Linear equalization shows a clear trade-off between signal improvement and noise enhancement due to nulls in the channel transfer function.

The Viterbi algorithm (VA) [1] utilizes the principle of dynamic programming to achieve maximum-a-posteriori (MAP) detection of a symbol sequence with a finite symbol alphabet passing through a channel with known transfer characteristic. The resulting nonlinear recursive filter shows a superior performance over linear techniques at the expense of an increased detector complexity. Other applications are in the field of convolutional coding, speech and text recognition [1].

It is the objective of this paper to extend those findings in communication theory to the two-dimen-sional problem of image restoration.

An Image Degradation Model in Form of a Discrete Markov Process

The aim of this chapter is to derive a causal Markov model for the two-dimensional spatial dispersion.

The underlying model for the VA is a finite state Markov process [1]. The probability of being in state $\underline{x}^{k+1}$ at time instant k+1 given all states from the past depends only on the previous state $\underline{x}^k$:

$$P(\underline{x}^{k+1}|\underline{x}^0,\underline{x}^1,\ldots,\underline{x}^k)=P(\underline{x}^{k+1}|\underline{x}^k). \qquad (2.1)$$

As in [1] we define a transition $\underline{\xi}^k$ at time k as a pair of states

$$\underline{\xi}^k \triangleq (\underline{x}^{k-1}, \underline{x}^k). \qquad (2.2)$$

The process is assumed to be observed in memoryless noise, namely there is a sequence $\underline{Z}$ of observations $\underline{Z} := [\underline{z}^1, \underline{z}^2,\ldots, \underline{z}^{N-n+1}]$ in which $\underline{z}^k$ depends probabilistically only on the transitions $\underline{\xi}^k$:

$$P(\underline{Z}|\underline{X})=P(\underline{Z}|\Xi)=\prod_{k=1}^{N-n+1} P(\underline{z}^k|\underline{\xi}^k). \qquad (2.3)$$

This is a very general model which includes, for example, time varying and nonlinear characteristics.

In this paper, however, we limit our description to a rather conservative model in order to clarify the main idea. The degradation model is assumed to be discrete. Topics dealing with the approximation of the continuous model through digitizing and sensor influences will not be considered in this paper.

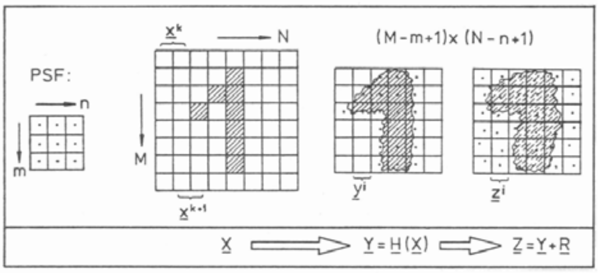

Fig . 2: Degradation example of a binary image

As shown in Fig. 2, the original image $\underline{X}$ of dimension M×N is distorted by a deterministic transfer function H with a finite dispersion length of m×n which results in a uniquely defined distorted image $\underline{Y}$ of dimension (M-m+1)×(N-n+l). The memoryless noise $\underline{R}$ is assumed to be additive:

$$\underline{Z} = H(\underline{X}) + \underline{R} \qquad (2.4)$$

To get a causal model we define the states $\underline{x}^k$ as sections of the original image $\underline{X}:= [\underline{x}^0, \underline{x}^1,\ldots, \underline{x}^{N-n+1}]$ of dimension M×N as:

$$\underline{x}^k = \begin{bmatrix} X_{1,k+1} & \cdots & X_{1,k+n-1} \\ X_{2,k+1} & \cdots & X_{2,k+n-1} \\ \vdots & & \vdots \\ X_{M,k+1} & \cdots & X_{M,k+n-1} \end{bmatrix}, \; k=0,1,\ldots,(N\text{-}n+1). \qquad (2.5)$$

and a transition as:

$$\underline{\xi}^k = \left[\begin{array}{c|c} & X_{1,k+n-1} \\ & X_{2,k+n-1} \\ \underline{x}^{k-1} & X_{3,k+n-1} \\ & \vdots \\ & X_{M,k+n-1} \end{array} \right], \; k=1,2,\ldots,(N\text{-}n+1). \qquad (2.6)$$

See for example Fig. 3 for the binary case with m=2, n=2, M=2, N=8. The pixel amplitudes are assumed to be elements of a finite alphabet of B admissible gray levels.

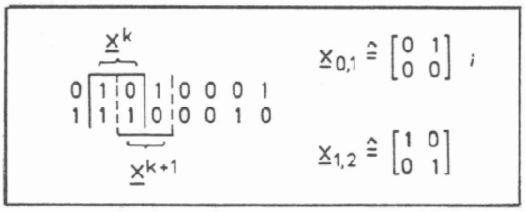

Fig. 3. The definition of a causal state sequence for the example of a low-dimensional binary image with spatial dispersion.

Under these assumptions, the degradation model may be described as M parallel shift registers with the states $\{\underline{x}^k\}$ as input and columns of the distorted image $\{\underline{y}^k\}$ as output sequence, corrupted by an additive noise sequence $\{\underline{r}^k\}$.

<u>Maximum-A-Posteriori Estimation Algorithm</u>

The problem of optimal restoration may now be stated as:

> Given a sequence of observations in form of sampled columns of the degraded image $\underline{Z} = \{\underline{z}^k\}$, find the state sequence $\underline{X} = \{\underline{x}^k\}$ which maximizes the a-posteriori probability

$$P(\underline{X}|\underline{Z}). \qquad (3.1)$$

It should be stressed that this criterion tries to optimally restore the whole image and not individual pixels.

Due to the Markov and memoryless properties the logarithm of this performance criterion may be decomposed into sums and the problem is equivalent to finding the shortest path through a decision tree with weights proportional to [l]:

$$\lambda(\underline{\xi}^k) = -\ln P(\underline{x}^{k+1}|\underline{x}^k) - \ln P(\underline{z}^k|\underline{\xi}^k). \qquad (3.2)$$

The VA now tracks the minimal path along a decision trellis with $B^{(n-1)\cdot M}$ states and B^M transitions per state and hence a total of the order of $(N\text{-}n+1)\cdot B^{n\cdot M} \approx \cdot N \cdot B^{n\cdot M}$ probability computations.

The VA may also be interpreted as an efficient technique to classify a degraded image in comparison with the exponentially growing number of all $B^{M\cdot N}$ possible images out of $\mathcal{X}$ in the observation space $\mathcal{Z}$. Fig. 4 shows a decision trellis with 16 states and 4 transitions per state which can also be described as a nonlinear fast base-4 algorithm of dimension 16 [2].

If all original images in $\mathcal{X}$ are equally likely, the MAP criterion is equivalent to the maximum likelihood criterion.

The detection or classification performance in the observation space $\mathcal{Z}$ is dominated by the minimum distance between all possible signals in relation to the noise level, a problem which also turns out to be nonlinear. Ref. [2] shows how to calculate the minimum distance with the VA itself and an increased basis.

The path metric in Eq. (3.2) results in a simple Euclidean metric if white Gaussian noise is assumed. It is possible to take into account nonwhite noise by using a prewhitening filter [1] or by changing the metric [3].

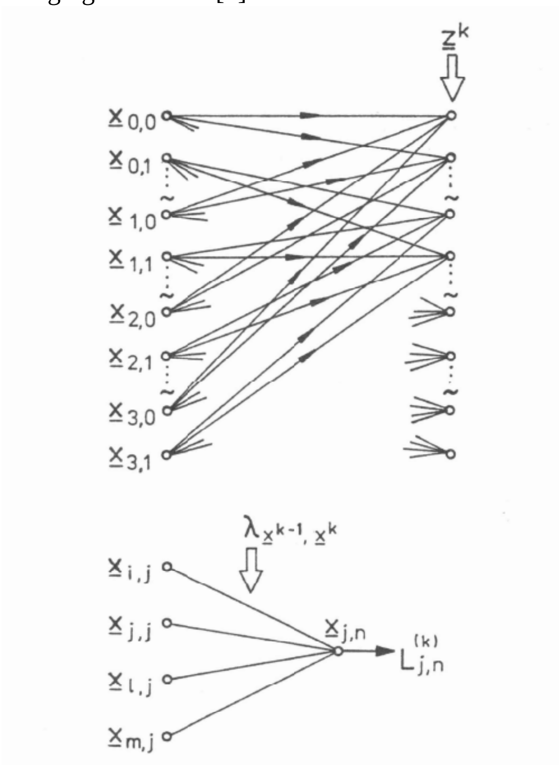

Fig. 4. Decision trellis, derived from the example in Fig. 3 with a base-4 algorithm and 16 possible states

The possibility to include constraints into the detector is shown in [2] for the case of code restrictions. This idea could also be used to incorporate given constraints of the original image into the restoration algorithm.

Examples

Typical examples for images with a-priori available amplitudes in form of a binary alphabet are facsimile pictures like weather charts or printed matter. The following two examples show restoration results for degraded binary test images with a one- and two-dimensional linear distortions respectively and white Gaussian noise of unknown intensity. The signal to noise ratio is related to the amplitude of a white pixel in the original image. As a comparison the results of linear inverse filtering using the pseudoinverse [4] are given.

Fig. 5 shows the restoration of an image degraded by a uniform motion blur over 7 pixels with the following one-dimensional pulse response:

$$h=1/7[1,1,1,1,1,1,1] \ . \qquad (4.1)$$

In this case the rows are decoupled and consequently a less complex one-dimensional base-2 algorithm with $2^6 = 64$ states could be used. It is worth mentioning that the algorithm was started in state zero.

Fig. 6 contains the filtering results for a two-dimensional pulse response of dimension 2x2 in the following form:

$$\underline{H}=\begin{bmatrix} 0.5 & 0.2 \\ 0.2 & 0.1 \end{bmatrix} . \qquad (4.2)$$

The size of the original image is of dimension (M=9)×(N=11). The algorithm contains $2^9 = 512$ possible states with 512 transitions per state. It should be mentioned that some additional savings can be found for the calculation of all transition metrics within one layer.

A complete closure of the eye diagram in both examples makes it impossible to restore the images using local decisions in form of simple amplitude clippings.

The figure ·of the second example also shows the clipped inverse filtering results. This nonlinear postfiltering is given as a comparison and could be interpreted as a simple way to take into account the a-priori knowledge of binary valued originals.

Conclusion

The application of the Viterbi algorithm to image restoration subject to the constraint of a finite amplitude alphabet of the original image gives substantial improvements compared to linear filtering. The application to high-dimensional images and a wide spread function can be a problem with respect to the computational complexity. To alleviate this difficulty it can be feasible to apply the algorithm to subsections of the image with an expected null gap at the boundary such as sufficient blank lines in text etc. It appeals to extend the proposed solution to more complicated models. A nonlinear spread function may be considered straightforwardly without affecting the principle solution (see [5]).

Acknowledgement

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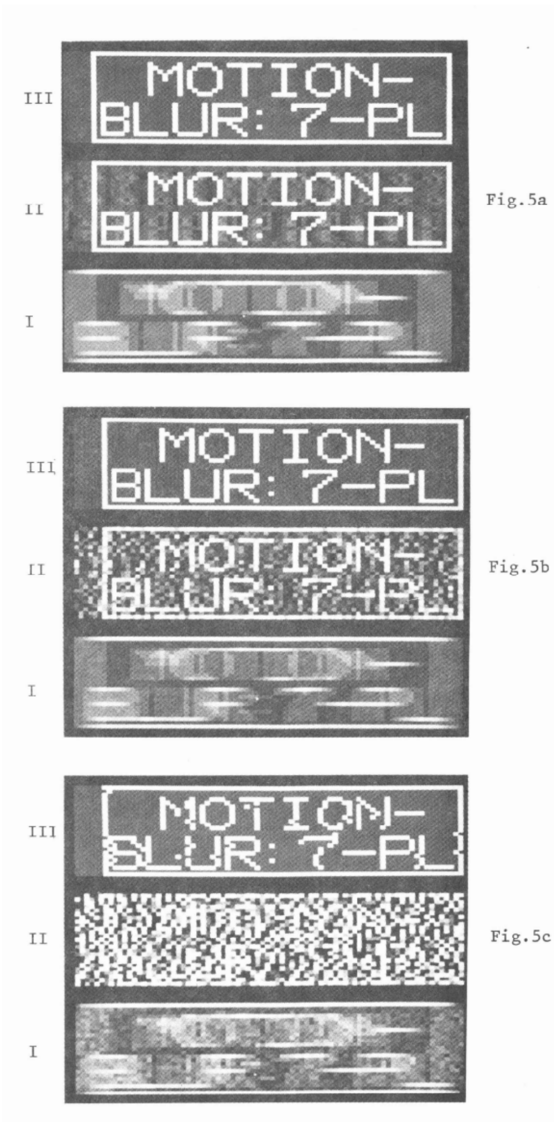

Fig. 5: Restoration of motion blur
I: Distorted Images
II: Result of pseudoinverse filtering
III: Restoration with Viterbi algorithm
a) no noise, b) SNR = 33, c) SNR = 15

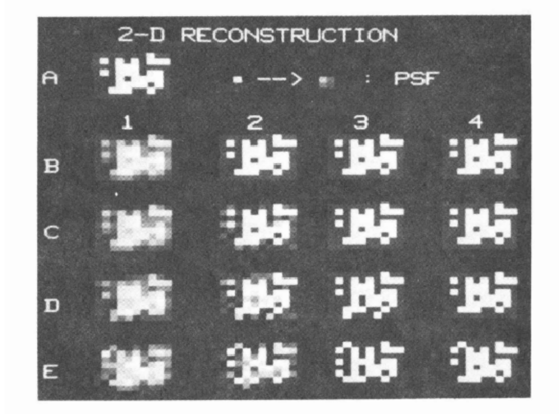

Fig. 6: Restoration of a two-dimensional dispersion
A: Original image and pulse response
1: Distorted images
2: Pseudoinverse filtering
3: Pseudoinverse filtering clipped
4: Viterbi algorithm
B: no noise
C: SNR 30
D: SNR 15
E: SNR 10

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