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Übung zur Vorlesung
Algorithmen zur digitalen Bildverarbeitung I

Blatt 4+5: Hilbert Space, Best Approximation, Pseudoinverse

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Aufgabe 1:

Consider an arbitrary vector $\mathbf{x}$ of the Euclidean vector space $\mathbb{R}^n$. Given a subspace $U \subset \mathbb{R}^n$ with basis $\{\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_m\}$. Derive the best approximation $\mathbf{y} \in U$ of $\mathbf{x}$ in U by estimating the expansion coefficients $\alpha_0, \alpha_1 \ldots \alpha_m$ such that

$$E = ||\mathbf{x} - \mathbf{y}||^2 = ||\mathbf{x} - \sum_{i=0}^{m} \alpha_i \mathbf{e}_i||^2 \stackrel{!}{=} \min \tag{1}$$

without using the projection theorem !

Aufgabe 2:

Compute the Fourier transformations of the following functions:

$$g_1(t) = \mathrm{rect}(t/T_0) = \begin{cases} 1, & |t| < T_0/2 \\ 1/2, & |t| = T_0/2 \\ 0, & |t| > T_0/2 \end{cases}$$

$$g_2(t) = \delta(t)$$

$$g_3(t) = 1$$

$$g_4(t) = \cos(2\pi f_0 t)$$

$$g_5(t) = \sin(2\pi f_0 t)$$

$$g_6(t) = \sum_{i=-\infty}^{\infty} \delta(t - iT_0)$$

$$g_7(t) = \sigma(t) = \begin{cases} 0, & x \le 0 \\ 1, & x > 0 \end{cases}$$

[**Comments:**] The *Dirac-Impulse* $\delta(t)$ is defined by the following property:

$$\delta(t) = \langle \delta, f \rangle = \int_{-\infty}^{\infty} \delta(t) f(t) dt = f(0)$$

Shifting property of the Fourier Transformation:

$$g(t - t_0) \circ\!\!-\!\!\bullet\ \widetilde{g}(f) e^{-2\pi i a t_0 f}$$

Modulation property of the Fourier Transformation:

$$g(t) e^{-2\pi i a f_0 t} \circ\!\!-\!\!\bullet\ \widetilde{g}(f - f_0)$$

Aufgabe 3:

Consider the linear vector space V of continuous functions defined on the interval $[-1, 1]$.

1. Show that using the inner product

$$< x(t), y(t) >:= \int_a^b x(t) \cdot y^*(t) dt,$$

and the norm

$$||x(t)|| := \sqrt{< x(t), x(t) >},$$

the vector space V is not forming a Hilbert space.
[**Hint:**] Make use of the following sequence

$$x_1(t),\ x_2(t),\ \ldots,\ x_k(t),\ \ldots$$

with

$$x_k(t) = \begin{cases} 1, & \frac{1}{k} \leq t \leq 1 \\ \frac{kt+1}{2}, & -\frac{1}{k} \leq t < \frac{1}{k} \\ 0, & -1 \leq t < -\frac{1}{k} \end{cases}$$

Aufgabe 4:

Given the following measurements (x_i, y_i):

x_i	-2	-1	0	1	2
y_i	44	11	-2	5	32

Estimate the parameter vector $\mathbf{c} = \left(c_0, c_1, c_3\right)^T$ for the function

$$p(x) = c_0 + c_1 x + c_2 x^2$$

approximating the measurements by minimising the error:

$$E = \min_{c_i} \sum_i (p(x_i) - y_i)^2$$

[**Hints:**]

1. Specify the equation $\mathbf{Ac} - \mathbf{y} = \mathbf{r}$, where $\mathbf{r}$ is the error vector.
2. Derive the equation $\mathbf{A}^T\mathbf{Ac} - \mathbf{A}^T\mathbf{y} = 0$ by optimising the error $E = ||\mathbf{r}||^2$.
3. Estimate the parameter vector $\mathbf{c}$ using the pseudo inverse.
4. Determine the error vector $\mathbf{r}$.

Aufgabe 5:

Theorem: For any $n \times n$ matrix $\mathbf{A}$ exist unitary matrices $\mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n]$ and $\mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]$ such that

$$\mathbf{U}^*\mathbf{A}\mathbf{V} = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n) := \mathbf{\Sigma}$$

with

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0.$$

1. Show that the decomposition $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^*$ exists.
2. Suppose that $\mathrm{rank}(\mathbf{A}) = r$, which means that

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > \sigma_{r+1} = \sigma_{r+2} = \cdots = \sigma_n = 0,$$

   Find the range $R(\mathbf{A})$ and null space $N(\mathbf{A})$ of $\mathbf{A}$ and $\mathbf{A}^*$ respectively.
3. Show that $\mathbf{A} : R(\mathbf{A}^*) \to R(\mathbf{A})$ is bijective.
4. Define $\mathbf{A}^+$ such that

$$\begin{cases} \mathbf{A}^+ u_i = \frac{1}{\sigma_i} v_i & (1 \leq i \leq r) \\ \mathbf{A}^+ u_i = 0 & (r < i \leq n) \end{cases},$$

   show that for any vector $y \in C^n$, $x_0 = \mathbf{A}^+ y$ satisfies
   (a) $\|\mathbf{A}x_0 - y\| \leq \|\mathbf{A}x - y\| \qquad \forall x \in C^n$
   (b) if $\|\mathbf{A}x_0 - y\| = \|\mathbf{A}x - y\|$, then $\|x_0\| < \|x\|$.
5. Find the expression of $\mathbf{A}^+$ and show that it satisfies the *Moore-Penrose* conditions:
   (a) $\mathbf{A}\mathbf{A}^+\mathbf{A} = \mathbf{A}$
   (b) $\mathbf{A}^+\mathbf{A}\mathbf{A}^+ = \mathbf{A}^+$
   (c) $(\mathbf{A}^+\mathbf{A})^* = \mathbf{A}^+\mathbf{A}$
   (d) $(\mathbf{A}\mathbf{A}^+)^* = \mathbf{A}\mathbf{A}^+$,

[**Comments:**] The theorem for non-square matrices: For any $m \times n$ matrix $\mathbf{A}$, there exist unitary matrices $\mathbf{U} = [u_1, u_2, \ldots, u_m]$ and $\mathbf{V} = [v_1, v_2, \ldots, v_n]$ such that $\mathbf{\Sigma} = \mathbf{U}^*\mathbf{A}\mathbf{V}$ is a $m \times n$ diagonal matrix with real, non-negative elements $\sigma_i,\ i = 1, 2, \ldots, \min(m, n)$ in descending order:

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\min(m,n)} \geq 0.$$

Similar analysis can be applied to the non-square case.