ALBERT-LUDWIGS-UNIVERSITÄT FREIBURG
INSTITUT FÜR INFORMATIK
Lehrstuhl für Mustererkennung und Bildverarbeitung
Prof. Dr.-Ing. Hans Burkhardt

Georges-Köhler-Allee
Geb. 052, Zi 01-029
D-79110 Freiburg
Tel. 0761 - 203 - 8260

Übung zur Vorlesung
Algorithmen zur digitalen Bildverarbeitung I

Blatt 3: Linear Mappings, Sum, Direct Sum, Projection onto Subspace

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Aufgabe 1:

Given the following three vectors:

$$\mathbf{c}_0 = \left(\tfrac{1}{\sqrt{2}}, 0, \tfrac{i}{\sqrt{2}}\right)^T \qquad \mathbf{c}_1 = \left(\tfrac{-1}{\sqrt{2}}, 0, \tfrac{i}{\sqrt{2}}\right)^T \qquad \mathbf{c}_2 = \left(0, 1, 0\right)^T \tag{1}$$

a)

Show that $\mathbb{C}^3 = \mathrm{S}(\mathbf{c}_0) \oplus \mathrm{S}(\mathbf{c}_1, \mathbf{c}_2) = \mathrm{S}(\mathbf{c}_1) \oplus \mathrm{S}(\mathbf{c}_0, \mathbf{c}_2)$. $\mathrm{S}(\mathbf{v}_1, \ldots \mathbf{v}_n)$ denotes the vector space spanned by the vectors $\mathbf{v}_1, \ldots \mathbf{v}_n$.

b)

Find the matrix $\mathbf{P}$ representing the projection from $\mathbb{C}^3$ onto $\mathrm{S}(\mathbf{c}_0)$ along $\mathrm{S}(\mathbf{c}_1, \mathbf{c}_2)$ as well as the matrix $\mathbf{Q}$ representing the projection from $\mathbb{C}^3$ onto $\mathrm{S}(\mathbf{c}_1, \mathbf{c}_2)$ along $\mathrm{S}(\mathbf{c}_0)$.

c)

Show that the projections $\mathbf{P}$ and $\mathbf{Q}$ are orthogonal projections.

Aufgabe 2:

Suppose a linear mapping $F : \mathcal{X} \to \mathcal{Y}$ is bijective. Show that the inverse mapping $F^{-1} : \mathcal{Y} \to \mathcal{X}$ is also linear.

Aufgabe 3:

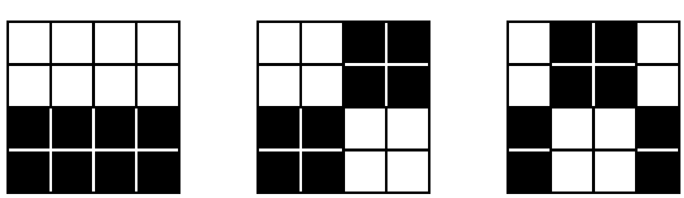
Let $\mathbf{V}$ be the vector space of functions with basis $B = \{\sin t, e^{7.5t}, \cos t\}$, and let $D : \mathbf{V} \to \mathbf{V}$ be the differential operator defined by $Df(t) = \frac{\partial f(t)}{\partial t}$. Compute the matrix representing D in the basis B.

Aufgabe 4:

Show that a linear mapping $P : \mathcal{X} \to \mathcal{X}$ is a projection if, and only if: $P^2 = P$;

Aufgabe 5:

Given three orthogonal images spanning a subspace of the vector space $\mathbb{C}^4 \times \mathbb{C}^4$:



$\mathbf{X}_1$ $\mathbf{X}_2$ $\mathbf{X}_3$

■ has value -1 □ has value 1

1. Find the orthogonal projection of the following image:

$$X = \begin{bmatrix} 5 & 4 & -1 & -5 \\ 3 & 2 & 0 & 1 \\ -2 & 0 & 3 & 2 \\ -4 & -2 & 5 & 6 \end{bmatrix}$$

   onto the subspace $\mathrm{S}(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$.

2. Find the error vector created by the projection onto the subspace, and show that this error vector is orthogonal to the projected image.

Hint:

For the orthogonal projection $\mathbf{y} \in Y$ of a vector $\mathbf{x} \in X \supset Y$ the following equation holds:

$$\mathbf{y} = \sum_i \langle \mathbf{x}, \mathbf{e}_i \rangle \, \mathbf{e}_i,$$

where $\{\mathbf{e}_i\}$ is an orthonormal basis of Y.